Segmentation using Level Set Methods

Geodesic and Region Based Active Contours

Pavel Matula

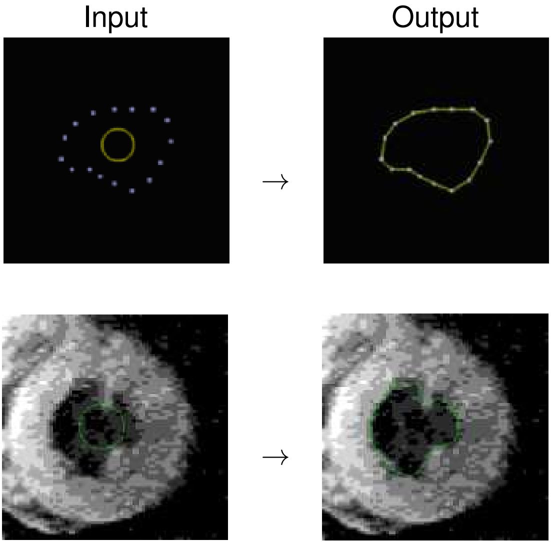
21. května 2012

Contents

Contents

Contents

Snakes, Motivation



Source: `http://www.iacl.ece.jhu.edu/static/gvf/`

Segmentation using Active Contours

- Two views on active contour segmentation
  1. curve deformation is driven by external and internal forces in order to capture important structures (usually edges) in an image.
  2. One searches for a curve (image partition) with minimal energy (defined by a functional)
- There is often a close mathematical connection between these two views, however
  - Not all deformational rules imply reasonable and understandable energy minimisation problem.
  - Some energy minimisation problems are practically unsolveable.

Basic Model

- Curve $\mathcal{C}$ is (1D) continuous set of points.
- Energy of the curve can be defined by a functional, e.g.

$$E(\mathcal{C}) = \int_{\mathcal{C}} E_{int}(\mathcal{C}(q)) + E_{ext}(\mathcal{C}(q))dq$$

- Necessary condition for energy minima is described by Euler-Lagrange equation derived from energy functional. It is partial differential equation representing force balance.

$$0 = F_{int}(\mathcal{C}) + F_{ext}(\mathcal{C})$$

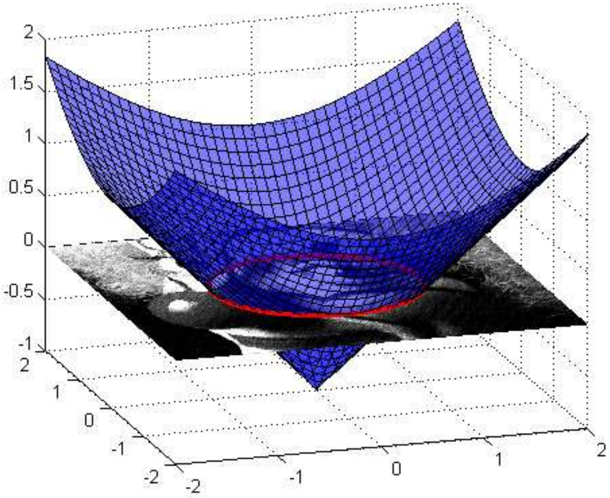
- It is often not possible to directly find solution of Euler-Lagrange equation. In general, initial curve $\mathcal{C}_0 = \mathcal{C}(t = 0)$ is deformed according to evolution equation

$$\frac{\partial \mathcal{C}}{\partial t} = F_{int}(\mathcal{C}) + F_{ext}(\mathcal{C})$$

  toward a stable state where the curve does not change in time, i.e., $\frac{\partial \mathcal{C}}{\partial t} = 0$ and therefore we achieve force balance.
- Convex functionals guarantee solution uniqueness.

Contents

Level Set Methods



Easy Normal Computation

- Let $C = \{x : u(x) = 0\}$ be a curve ($x \in \mathbb{R}^2$) or a surface ($x \in \mathbb{R}^3$).
- Then the unit normal vector at the point x is defined by

$$n(x) = \frac{\nabla u}{|\nabla u|}$$

Easy Curvature Computation

- The curvature is defined by (= mean curvature for $\mathbb{R}^3$)

$$\kappa(x) = \nabla \cdot n(x) = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right),$$

  as the divergence of the direction of the gradient of u.
- The curvature can be computed in terms of the partial derivatives as:

$$\kappa(x) = (u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx})/|\nabla u|^3$$

$$\begin{aligned}\kappa(x) =&(u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx} + u_x^2 u_{zz} - 2u_x u_z u_{xz} \\ &+ u_z^2 u_{xx} + u_y^2 u_{zz} - 2u_y u_z u_{yz} + u_z^2 u_{yy})/|\nabla u|^3\end{aligned}$$

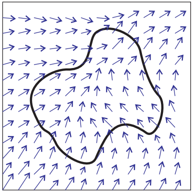
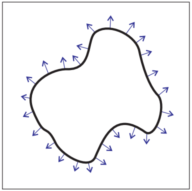
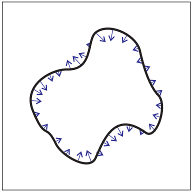
General Procedure

- Embed initial curve into implicit function u as its zero level set. Often, signed distance function is used.
- Evolve the function u according to the evolution equation.

$$u_t = \beta|\nabla u|$$

  with appropriate forces β.
- All contours (including the zero contour are evolved).

Three Types of Motion

external velocity	normal motion	curvature motion
		

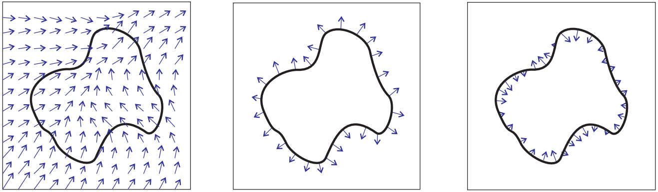
$$u_t = \quad -V \cdot \nabla u \quad -a|\nabla u| \quad +\epsilon\kappa|\nabla u|$$

It is discretized with

$$u_{ij}^{n+1} = u_{ij}^{n} - \tau \left[\begin{array}{c} \left[\begin{array}{c} \max(w_{ij}^{n},0)D_{ij}^{-x} + \min(w_{ij}^{n},0)D_{ij}^{+x} \\ +\max(v_{ij}^{n},0)D_{ij}^{-y} + \min(v_{ij}^{n},0)D_{ij}^{+y} \end{array} \right] + \\ +[\max(a,0)\nabla^{+} + \min(a,0)\nabla^{-}] \\ \\ -[\epsilon K_{ij}^{n}\sqrt{(D_{ij}^{0x})^2 + (D_{ij}^{0y})^2}] \end{array} \right].$$

Three Types of Motion

external velocity | normal motion | curvature motion

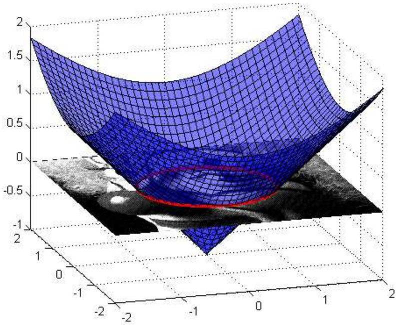


$$u_t = \quad -V \cdot \nabla u \quad -a|\nabla u| \quad +\epsilon\kappa|\nabla u|$$

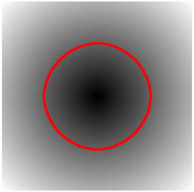
It is discretized with

$$u_{ij}^{n+1} = u_{ij}^{n} - \tau \cdot \left[\text{Velocity}(V) + \text{Normal}(a) - \text{Curvature}(\epsilon)\right].$$

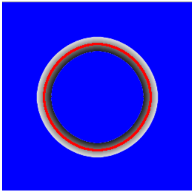
Level Set Methods, Evolution of One Contour



Whole domain



Narrow band



- We can compute the evolution in a narrow band around the zero level set only!

Contents

Contents

Geodesic Active Contours

- Caselles et.al. 1995, Kichenassamy et al. 1995
- Curve with minimal geodesic length is searched.

$$E_{GAC}(\mathcal{C}) = \int_0^1 g(|\nabla G_\sigma * I(C(q))|)|C'(q)|dq$$

where $g : [0, \infty) \rightarrow \mathbb{R}^+$ is a strictly decreasing function such that $g(r) \rightarrow 0$ as $r \rightarrow \infty$, e.g.,

$$g(|\nabla G_\sigma * I(x, y)|) = \frac{1}{1 + |\nabla G_\sigma * I(x, y)|}$$

- The curve is attracted by image edges, where the weight $g(|\nabla G_\sigma * I(C(q))|)$ is small.
- Energy functional is not convex and therefore there are several local minima.

Geodesic Active Contours, Evolution Equation

- Evolution equation (i.e. Euler-Lagrange equation with $\frac{\partial \mathcal{C}}{\partial t}$ on the left side) for geodesic active contours is

$$\frac{\partial \mathcal{C}}{\partial t} = g\kappa n - (\nabla g \cdot n)n$$

  where n is curve normal (vector) and κ is curvature (scalar)
- This equation can be rewritten in level-set framework. The curve is embeded in signed distance function u (with evolution speed $\nu = g\kappa - \nabla g \cdot n$). The evolution equation becomes

$$\frac{\partial u}{\partial t} = g\kappa|\nabla u| - \nabla g \cdot \nabla u$$

Geodesic Active Contours, Evolution Equation 2

- The evolution equation

$$\frac{\partial u}{\partial t} = g\kappa|\nabla u| - \nabla g \cdot \nabla u$$

contains curvature motion and velocity field motion. It is solved using level set methods.

- Often, one consider slightly different equation (with normal direction motion; corresponds to baloon force)

$$\frac{\partial u}{\partial t} = (c + \epsilon\kappa)g|\nabla u| + \beta\nabla P \cdot \nabla u$$

where $P = |\nabla G_\sigma * I|$, c is a constant and

$$g(|\nabla G_\sigma * I(x,y)|) = \frac{1}{1 + |\nabla G_\sigma * I(x,y)|^p}$$

ϵ, β, c, σ and p are parameters.

Contents

Geodesic Active Contours, Iterative Scheme

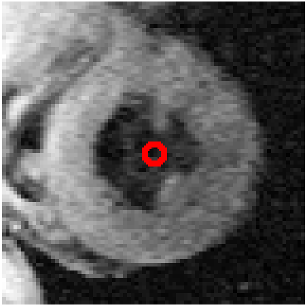
- The equation

  $$u_t = (c + \epsilon\kappa)g|\nabla u| + \beta\nabla P \cdot \nabla u$$

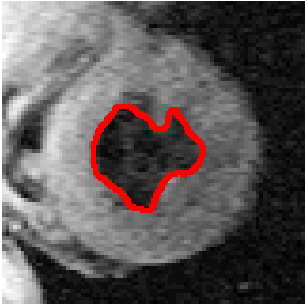
  consists of three types of motion we have discussed before:
  - normal direction motion with speed $cg(|\nabla G_\sigma * I(x, y)|)$
  - curvature motion multilied by a factor $\epsilon g(|\nabla G_\sigma * I(x, y)|)$
  - external velocity field motion given by $\beta\nabla P$.
- Therefore, we have

  $$u_{ij}^{k+1} = u_{ij}^{k} + \tau \cdot [\text{Normal}(cg) + \text{Curvature}(\epsilon g) + \text{Velocity}(\beta\nabla P)].$$

Geodesic Active Contours: Example



Initial contour



Result

$\tau = 0.25, c = 1, \epsilon = 0.5,$

$\beta = 0.5, p = 2, \sigma = 2.0$

Contents

Region Based Active Contours: Motivation

- Region interior was not considered in previously discussed active contours!
- No region homogeneity was required (image structures under the curve are important for solution only).

Contents

Mumford-Shah Functional (1985)

- Functional formulation of segmentation.
- One seeks segmentation (u, K) of some image $f : \Omega \to \mathbb{R}$ as the minimiser of

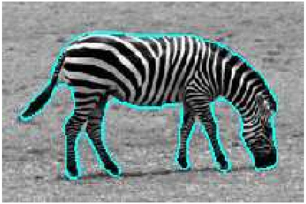
$$E_{MS}(u, K) := \lambda \int_{\Omega} (u - f)^2 dx + \beta \int_{\Omega \setminus K} |\nabla u|^2 dx + \mu |K|$$

  where $u(x)$ is smoothed version of $f(x)$, and K represents its edges
- First term penalises deviations from original image f
- Second term penalises variations within each segment
- Third term penalises the edge length $|K|$
- Mathematically very difficult (we seek edges K as well as image u)
- Several approximations of functional E_{MS} were proposed to overcome these difficulties.
- Unique solution does not exist in general.

Contents

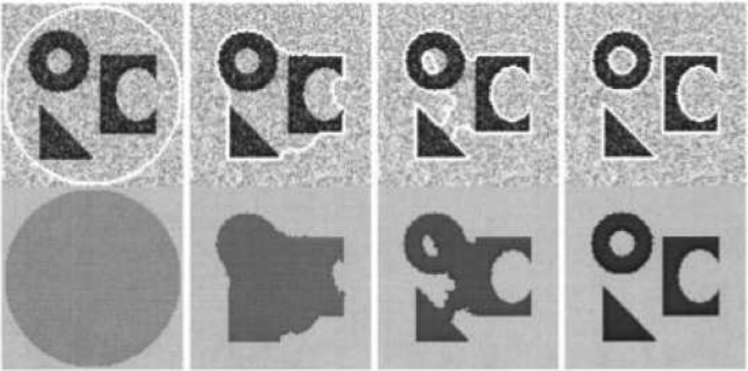
Active-Contours Without Edges

- Chan and Vese 2001
- **Given**: Image $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$.
- **Goal**: Segmentation of Ω into two regions (possibly disconnected)
- Curve evolution is based on region information (but not on edges)
- Can be extended to segment color and textured images.



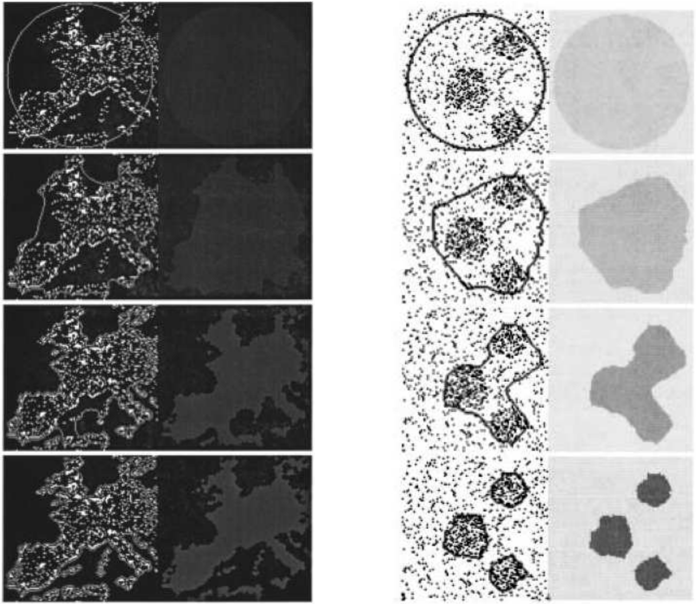
source: http://www.mia.uni-saarland.de/IS_Research.shtml

Active Contours Without Edges: Example 1



Source: Chan, Vese, Active Contours Without Edges, 2001

Active Contours Without Edges: Example 2



Source: Chan, Vese, Active Contours Without Edges, 2001

Active-Contours Without Edges: Idea

- Regions are separated by a curve $\mathcal{C}$.
- In each region, a constant grey-value is supposed to approximate the image.
- **Data term** penalises the deviation from the piecewise constant approximation of the input image
- **Regularity term** impose regularity constraints for the curve (requires curve of minimal length).

Active-Contours Without Edges: Functional

- Let $f(x) : \Omega \to \mathbb{R}$ is the input image.
- Let $\mathcal{C}$ be the boundary between two regions Ω_1 and Ω_2. $\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{C}$
- Chan-Vese functional is defined

$$E_{CV}(\mathcal{C}, c_1, c_2) = \mu L(\mathcal{C}) + \lambda_1 \int_{\Omega_1} |f(x) - c_1|^2 dx + \lambda_2 \int_{\Omega_2} |f(x) - c_2|^2 dx$$

  where $\mu \geq 0, \lambda_1, \lambda_2 \geq 0$ are given fixed parameters (weights), $L(\mathcal{C})$ denotes the length of $\mathcal{C}$ and c_1 and c_2 are mean intensity values of two distinct regions
- We minimize the functional with respect to c_1,c_2, and $\mathcal{C}$.

Equivalence with Mumford-Shah Functional

- Chan-Vese functional:

$$E_{CV}(\mathcal{C}, c_1, c_2) = \mu L(\mathcal{C}) + \lambda_1 \int_{\Omega_1} |f(x) - c_1|^2 dx + \lambda_2 \int_{\Omega_2} |f(x) - c_2|^2 dx$$

- Mumford-Shah functional:

$$E_{MS}(u, K) := \lambda \int_{\Omega} (u - f)^2 dx + \beta \int_{\Omega \setminus K} |\nabla u|^2 dx + \mu |K|$$

- Equivalence for piecewice constant approximations u (the second term in MS functional is therefore zero) and edge set K that separates Ω only into two regions (and $\lambda = \lambda_1 = \lambda_2$).

Contents

Active-Contours Without Edges: Implementation

- Level set formulation: The curve $\mathcal{C}$ is represented as a zero level set of a continuous function $u : \Omega \to \mathbb{R}$. We get

$$
\begin{aligned}
E_{CV}(\mathcal{C}, c_1, c_2) = E_{CV}(u, c_1, c_2) = \\
&= \mu \int_\Omega |\nabla H(u(x))| dx \\
&+ \lambda_1 \int_\Omega (f(x) - c_1)^2 H(u(x)) dx \\
&+ \lambda_2 \int_\Omega (f(x) - c_2)^2 (1 - H(u(x))) dx
\end{aligned}
$$

where $H(u)$ is Heaviside function:

$$
H(u) = \begin{cases} 1 & \text{for } u \geq 0 \\ 0 & \text{for } u < 0 \end{cases}
$$

Evolution Equation

- Minimization of CV functional leads to the following evolution equation:

$$\frac{\partial u}{\partial t} = \delta_\epsilon(u)\left[\mu\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right) - \lambda_1(f-c_1)^2 + \lambda_2(f-c_2)^2\right]$$

where

$$c_1 = \frac{\int_\Omega f(x)H(u(x))dx}{\int_\Omega H(u(x))dx}, \quad c_2 = \frac{\int_\Omega f(x)(1-H(u(x)))dx}{\int_\Omega (1-H(u(x)))dx}$$

and δ_ϵ is regularized Dirac function. It is a derivative of

$$H_\epsilon(u) = \frac{1}{2}(1 + \frac{2}{\pi}tan^{-1}\left(\frac{u}{\epsilon}\right)).$$

- Notice, the first term is the curvature motion applied to u.
- u is changed only within narrow band where $\delta_\epsilon \neq 0$.

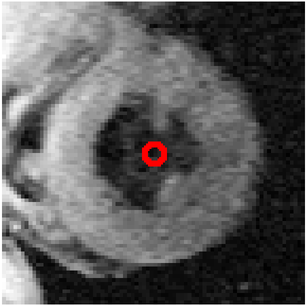
Discrete Evolution Equation

The discrete evolution equation is:

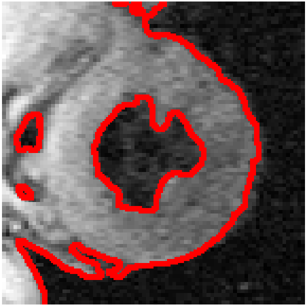
$$u_{ij}^{k+1} = u_{ij}^{k} + \tau\delta_{\epsilon}(u_{ij}^{k}) \cdot \left[\text{Curvature}(\mu) - \lambda_1(f_{ij} - c_1)^2 + \lambda_2(f_{ij} - c_2)^2\right]$$

where c_1 and c_2 are average intensities of foreground and background at time $k\tau$ and Curvature(μ) is the curvature term applied at u_{ij}.

Example 1



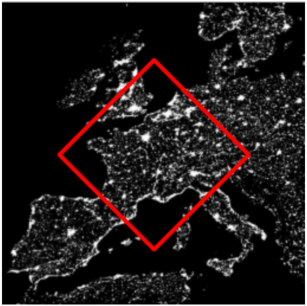
Initial contour



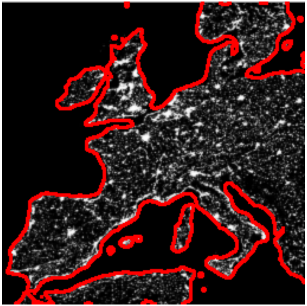
Result

$\tau = 0.5, \lambda_1 = \lambda_2 = 0.05,$
$\epsilon = 10, \mu = 10$

Example 2



Initial contour



Result

$\tau = 0.5, \lambda_1 = 1, \lambda_2 = 0.02,$
$\epsilon = 10, \mu = 10$

Contents

Part I: Summary

- We discussed active contours in the context of level set methods.
- Geodesic active contours seeks curve with minimal geodesic length.
- Mumford-Shah functional is a general variational formulation of segmentation. It is mathematically very difficult.
- Active contours without edges are region based segmentation approach. Considers homegenity of regions instead of edges. It is a special case of MS functional.