Affine Moment Invariants

We will suppose the affine moment invariant has form of a polynomial of moments

$$I = (\sum_{j=1}^{n_t} c_j \prod_{\ell=1}^{r} \mu_{p_{j\ell},q_{j\ell}})/\mu_{00}^{r+w} \ . \tag{1}$$

Cayley – Aronhold differential equation

$$\sum_p \sum_q p\mu_{p-1,q+1}\frac{\partial I}{\partial \mu_{pq}} = 0 \ . \tag{2}$$

Graph Method

Let us denote

$$C_{12} = x_1y_2 - x_2y_1.$$

After an affine transform it holds $C'_{12} = J \cdot C_{12}$, it means C_{12} is a relative affine invariant. It has also geometric meaning as the oriented double area of the triangle, whose one vertex is the origin of the coordinate system (centroid of the image f) and two other vertices are points (x_1, y_1) and (x_2, y_2).

More precisely, having r points ($r \geq 2$) we define functional I depending on r and on non-negative integers n_{kj} as

$$I(f) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \prod_{k,j=1}^{r} C_{kj}^{n_{kj}} \cdot \prod_{i=1}^{r} f(x_i, y_i)dx_i dy_i. \tag{3}$$

For $r = 3$ and $n_{12} = 2, n_{13} = 2, n_{23} = 0$ we get

$$\begin{aligned} I(f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1y_2 - x_2y_1)^2(x_1y_3 - x_3y_1)^2 f(x_1,y_1)f(x_2,y_2)f(x_3,y_3)dx_1dy_1dx_2dy_2dx_3dy_3 \\ &= m_{20}^2m_{04} - 4m_{20}m_{11}m_{13} + 2m_{20}m_{02}m_{22} + 4m_{11}^2m_{22} \\ &\quad -4m_{11}m_{02}m_{31} + m_{02}^2m_{40}. \end{aligned} \tag{4}$$

Each invariant generated by formula (3) can be represented by a planar connected graph, where each point (x_k, y_k) corresponds to one node and each cross-product C_{kj} corresponds to one edge of the graph. If $n_{kj} > 1$, the respective term $C_{kj}^{n_{kj}}$ corresponds to n_{kj} edges connecting k-th and j-th nodes. Thus, the number of nodes equals the degree of the invariant and the total number of the graph edges equals the weight w of the invariant. From the graph one can also learn about the orders of the moments the invariant is composed from and about its structure. The number of edges originating from each node equals the order of the moments involved. The corresponding graph to (4) is shown in Fig. 1.

Tensors

We can imagine the tensor as an r-dimensional array of numbers that expresses coordinates of some geometric figure in n-dimensional space together with rule, how they change in affine transform of the space. The rules can be two, covariant and contravariant.

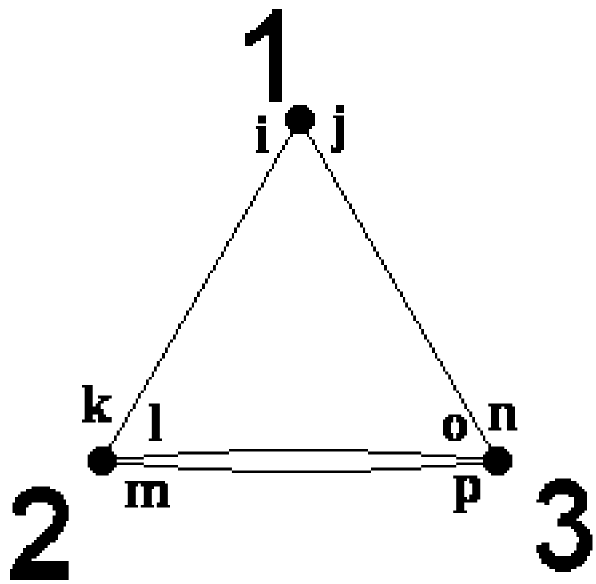

Figure 1: The graph corresponding to the invariant

A *contravariant vector* is given by n numbers x^1, x^2,... , x^n, that are transformed

$$x^i = \sum_{\alpha=1}^{n} p^i_\alpha \hat{x}^\alpha \ , \qquad i = 1, 2, \ldots, n \tag{5}$$

or in Einstein notation

$$x^i = p^i_\alpha \hat{x}^\alpha \ , \qquad i, \alpha = 1, 2, \ldots, n. \tag{6}$$

The i is not exponent, but upper index.

The affine transform p^i_α is in 2-dimensional case

$$\mathbf{A} = \begin{pmatrix} p^1_1 & p^1_2 \\ p^2_1 & p^2_2 \end{pmatrix} \ , \tag{7}$$

we assume its determinant is not zero. If we label the inverse transform

$$\mathbf{A}^{-1} = \begin{pmatrix} q^1_1 & q^1_2 \\ q^2_1 & q^2_2 \end{pmatrix} \ , \tag{8}$$

we can write

$$\hat{x}^\alpha = q^\alpha_i x^i \ , \qquad i, \alpha = 1, 2, \ldots, n. \tag{9}$$

A *covariant vector* is given by n numbers u_1, u_2,... , u_n, that are transformed

$$\hat{u}_\alpha = p^i_\alpha u_i \ , \qquad i, \alpha = 1, 2, \ldots, n. \tag{10}$$

A *covariant tensor* of order r is given by n^r numbers $a_{i_1, i_2, \ldots, i_r}$, that are transformed by affine transformation

$$\hat{a}_{\alpha_1, \alpha_2, \ldots, \alpha_r} = p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_r}_{\alpha_r} a_{i_1, i_2, \ldots, i_r} \ , \qquad i_1, i_2, \ldots, i_r, \alpha_1, \alpha_2, \ldots, \alpha_r = 1, 2, \ldots, n. \tag{11}$$

A *contravariant tensor* of order r is given by n^r numbers $a^{i_1,i_2,\dots,i_r}$, that are transformed by affine transformation

$$a^{i_1,i_2,\dots,i_r} = p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_r}_{\alpha_r} \hat{a}^{\alpha_1,\alpha_2,\dots,\alpha_r} \ , \qquad i_1, i_2, \dots, i_r, \alpha_1, \alpha_2, \dots, \alpha_r = 1, 2, \dots, n. \tag{12}$$

We can have also *mixed tensor* of the covariant order r_1, the contravariant order r_2 and the general order $r = r_1 + r_2$ with rule

$$\begin{gathered}\hat{a}^{\beta_1,\beta_2,\dots,\beta_{r_2}}_{\alpha_1,\alpha_2,\dots,\alpha_{r_1}} = q^{\beta_1}_{j_1} q^{\beta_2}_{j_2} \cdots q^{\beta_{r_2}}_{j_{r_2}} p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_{r_1}}_{\alpha_{r_1}} a^{j_1,j_2,\dots,j_{r_2}}_{i_1,i_2,\dots,i_{r_1}} \ , \\ i_1, i_2, \dots, i_{r_1}, j_1, j_2, \dots, j_{r_2}, \alpha_1, \alpha_2, \dots, \alpha_{r_1}, \beta_1, \beta_2, \dots, \beta_{r_2} = 1, 2, \dots, n.\end{gathered} \tag{13}$$

In multiplication of the tensors, there is a rule that we perform addition over indices that are used once as upper one and once as lower one and mere enumeration over indices used only once. It means mixed tensors with covariant order one and contravariant order one are multiplied by the same way like matrices

$$a^i_j b^j_k = c^i_k, \qquad i, j, k = 1, 2, \dots, n, \tag{14}$$

we perform addition over j and enumeration over i and k.

The *relative tensor* of the weight g is transformed

$$\begin{gathered}\hat{a}^{\beta_1,\beta_2,\dots,\beta_{r_2}}_{\alpha_1,\alpha_2,\dots,\alpha_{r_1}} = J^g q^{\beta_1}_{i_1} q^{\beta_2}_{i_2} \cdots q^{\beta_{r_2}}_{i_{r_2}} p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_{r_1}}_{\alpha_{r_1}} a^{i_1,i_2,\dots,i_{r_2}}_{i_1,i_2,\dots,i_{r_1}} \ , \\ i_1, i_2, \dots, i_{r_1}, j_1, j_2, \dots, j_{r_2}, \alpha_1, \alpha_2, \dots, \alpha_{r_1}, \beta_1, \beta_2, \dots, \beta_{r_2} = 1, 2, \dots, n.\end{gathered} \tag{15}$$

J is determinant of $\mathbf{A}$.

The moments do not behave in affine transform like tensors, but we can define *moment tensor* [1]

$$M^{i_1 i_2 \cdots i_r} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{i_1} x^{i_2} \cdots x^{i_r} f(x^1, x^2) dx^1 dx^2 \ , \tag{16}$$

where $x^1 = x$ and $x^2 = y$. $M^{i_1 i_2 \cdots i_r} = m_{pq}$ if p indices equal 1 and q indices equal 2. The behavior in affine transform

$$\begin{aligned}&M^{i_1 i_2 \cdots i_r} = |J| p^{i_1}_{\alpha_1} p^{i_2}_{\alpha_2} \cdots p^{i_r}_{\alpha_r} \hat{M}^{\alpha_1 \alpha_2 \cdots \alpha_r} \\ &\text{or} \\ &\hat{M}^{i_1 i_2 \cdots i_r} = |J|^{-1} q^{i_1}_{\alpha_1} q^{i_2}_{\alpha_2} \cdots q^{i_r}_{\alpha_r} M^{\alpha_1 \alpha_2 \cdots \alpha_r} \ , \qquad i_1, i_2, \dots, i_r, \alpha_1, \alpha_2, \dots, \alpha_r = 1, 2, \dots, n.\end{aligned} \tag{17}$$

It means the moment tensor is relative contravariant tensor with the weight $g = -1$. Sometimes it is called an oriented tensor, because of the factor $|J|$ instead of J.

For the explanation of the method of tensors, it is necessary to introduce a concept of unit polyvector. $\epsilon_{i_1 i_2 \dots i_n}$ is *covariant unit polyvector*, if it is skew-symmetric tensor over all indices and $\epsilon_{12\dots n} = 1$. The term *skew-symmetric* means that if we interchange two indices, the tensor element changes its sign and preserves its absolute value. In two dimensions, it means $\epsilon_{12} = 1$, $\epsilon_{21} = -1$, $\epsilon_{11} = 0$ and $\epsilon_{22} = 0$. In affine transform, it is changed

$$\hat{\epsilon}_{12\dots n} = J \epsilon_{12\dots n} \ , \tag{18}$$

i.e. it is relative affine invariant with weight 1. Then, if we multiply a proper number of moment tensors and unit polyvectors so the number of upper indices at the moment tensors equals the number of lower indices at polyvectors, we obtain one real number, relative affine invariant, e.g.

$$\begin{aligned} &M^{ij}M^{klm}M^{nop}\epsilon_{ik}\epsilon_{jn}\epsilon_{lo}\epsilon_{mp} = \\ &= 2[m_{20}(m_{21}m_{03} - m_{12}^2) - m_{11}(m_{30}m_{03} - m_{21}m_{12}) + m_{02}(m_{30}m_{12} - m_{21}^2)] \ . \end{aligned} \tag{19}$$

From an analysis of the computation, we can find this method is computationally equivalent to the method of graphs, each moment tensor corresponds to a node and each unit polyvector corresponds to an edge. The indices say, which edge connects which node. The graph corresponding to the invariant (19) is on Fig. 1.

Theorem

All affine moment invariants in the polynomial form (1) can be expressed as linear combinations of some invariants generated by the graph method

$$I^{(e)} = \sum_{P=1}^{n} c_P I_P^{(g)} \ . \tag{20}$$

Here $I^{(e)}$ is general affine moment invariant, e.g. generated as some solution of the Cayley-Aronhold differential equation, and $I_P^{(g)}$, $P = 1, \ldots n$ is set of invariants generated by the graph method with the same structure as $I^{(e)}$.

Proof

Gurevich deals with the proof of similar theorem in [2] or in Russian origin [3]. His theorem concerns with not only moments, but with all measurements of geometric objects, whose behavior in affine transform can be described by tensors. We have adapted a part of this proof related to the moments.

The invariant $I^{(e)}$ can be decomposed into a part of moments B and a part of coefficients K

$$I^{(e)} = K_{i_1 i_2 \ldots i_{2w}} B^{i_1 i_2 \ldots i_{2w}} \ , \tag{21}$$

where w is the weight of the invariant. The part B can be expressed as a product of moment tensors

$$B^{i_1 i_2 \ldots i_{2w}} = M^{i_1 i_2 \ldots i_{d_1}} M^{i_{d_1+1} i_{d_1+2} \ldots i_{d_1+d_2}} \cdots M^{i_{2w-d_r+1} i_{2w-d_r+2} \ldots i_{2w}} / m_{00}^{w+r} \ , \tag{22}$$

r is the degree of the invariant. The numbers d_1, d_2,... d_r is $I^{(e)}$ are orders of the moments.

The product of moment tensors in (22) contains all possible products of moments with the given structure, so the decomposition (21) is always possible. If some product of moments occurs several times (e.g. m times) in B, then the corresponding components of K must be multiplied by $1/m$. The invariants $I_P^{(g)}$ for each P can be decomposed to the part of coefficients and the part of moments as well, the part B is the same for all

$I_P^{(g)}$ and $I^{(e)}$, while the part of coefficients of $I_P^{(g)}$ can be expressed as a product of unit polyvectors. Then (20) can be rewritten as

$$K_{i_1 i_2 \dots i_{2w}} B^{i_1 i_2 \dots i_{2w}} = \sum_{P=1}^{(2w)!} c_P \epsilon_{\{i_1 i_2} \epsilon_{i_3 i_4} \cdots \epsilon_{i_{2w-1} i_{2w}\}_P} B^{i_1 i_2 \dots i_{2w}} \ , \tag{23}$$

where $\{i_1 i_2 \cdots i_{2w}\}_P$ means P-th permutation of the indices $i_1, i_2, \cdots, i_{2w}$. If the moments do not identically equal zero, then

$$K_{i_1 i_2 \dots i_{2w}} = \sum_{P=1}^{(2w)!} c_P \epsilon_{\{i_1 i_2} \epsilon_{i_3 i_4} \cdots \epsilon_{i_{2w-1} i_{2w}\}_P} \ . \tag{24}$$

It is important in it that the number of indices is twofold of the weight w of the invariant. The summation over all permutations of unit polyvector indices is not anything else than summation over all graphs generating invariants with the given structure. Note that ϵ_{ij} has nonzero value only if $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$, therefore $K_{i_1 i_2 \dots i_{2w}}$ can has nonzero value only if the number of indices with value 1 equals the number of indices with value two. It corresponds with fact that the sum of first indices equals the sum of the second indices in a term of an invariant. If we interchange 1's and 2's, the value of k would be multiplied by $(-1)^w$ – condition of symmetry.

Now, we multiply K by the corresponding number of contravariant unit polyvectors. The *contravariant unit polyvector* (in two dimensions $\epsilon^{i_1 i_2}$) has similar properties as covariant unit polyvector (it is skew-symmetric tensor over all indices and $\epsilon^{12\dots n} = 1$) except that it is multiplied as contravariant tensor, e.g.

$$\epsilon_{i_1 i_2} \epsilon^{i_1 i_2} = 2 \ . \tag{25}$$

Then we obtain from (24)

$$K_{i_1 i_2 \dots i_{2w}} \epsilon^{x_1 x_2} \epsilon^{x_3 x_4} \cdots \epsilon^{x_{2w-1} x_{2w}} = \sum_{P=1}^{(2w)!} c_P^* \delta_{\{i_1}^{x_1} \delta_{i_2}^{x_2} \cdots \delta_{i_{2w}\}_P}^{x_{2w}} \ , \tag{26}$$

where $c_P^* = 2^w c_P$ and $\delta_{i_2}^{i_1}$ is *Kronecker delta*, $\delta_{i_2}^{i_1} = 1$ if $i_1 = i_2$ and $\delta_{i_2}^{i_1} = 0$ if $i_1 \neq i_2$. The system of equations (26) has 2^{4w} equations for $(2w)!$ unknowns, but many of the equations are linearly dependent, the rank of the system was not increased. Denote it $((2w)! - s)$, where s is some integer greater than zero. Now take the system of equations

$$\sum_{P=1}^{(2w)!} \delta_{\{i_1}^{x_1} \delta_{i_2}^{x_2} \cdots \delta_{i_{2w}\}_P}^{x_{2w}} \lambda_P = 0 \ . \tag{27}$$

with unknowns λ_1, λ_2,..., $\lambda_{(2w)!}$. The matrices of the systems (26) and (27) are the same, therefore the rank of (27) is also $((2w)! - s)$. That is why the system (27) has s linearly independent solutions

$$\lambda_P = \lambda_P^\sigma, \ \sigma = 1, 2, \dots, s \ . \tag{28}$$

Now, we can add to the system (27) the s equations

$$\sum_{P=1}^{(2w)!} \lambda_P^\sigma \lambda_P = 0 \tag{29}$$

and obtain a system of $2^{4w}+s$ equations. Let the new connected system of equations (27) and (29) has some solution $\lambda_P=\lambda_P^0$, $P=1,2,\ldots,(2w)!$. This solution satisfies all the equations (27), therefore it must be a linear combination of the solutions λ_P^σ

$$\lambda_P^0=\sum_{\sigma=1}^{s}\alpha_\sigma\lambda_P^\sigma,\ P=1,2,\ldots,(2w)!\ . \tag{30}$$

The equations (29) must be satisfied for every λ_P, therefore they are satisfied for their arbitrary linear combinations and also for

$$\sum_{\sigma=1}^{s}\alpha_\sigma\sum_{P=1}^{(2w)!}\lambda_P^\sigma\lambda_P=0\ . \tag{31}$$

It can be rewritten by (30) in the form

$$\sum_{P=1}^{(2w)!}\lambda_P^0\lambda_P=0\ . \tag{32}$$

It must be satisfied for every λ_P thus also for λ_P^0

$$\sum_{P=1}^{(2w)!}(\lambda_P^0)^2=0\ , \tag{33}$$

i.e. $\lambda_1^0=\lambda_2^0=\cdots=\lambda_{(2w)!}^0=0$. It means the only solution of the connected system of equations (27) and (29) is zero and therefore its rank is $(2w)!$. To each of the solutions (28) corresponds a relation of the form

$$\sum_{P=1}^{(2w)!}\lambda_P^\sigma\delta_{\{i_1}^{x_1}\delta_{i_2}^{x_2}\cdots\delta_{i_{2w}\}_P}^{x_{2w}}=0\ . \tag{34}$$

Let $p_{x_1x_2\ldots x_{2w}}$ be an arbitrary tensor of covariance $2w$. Since

$$\delta_{i_1}^{x_1}\delta_{i_2}^{x_2}\ldots\delta_{i_{2w}}^{x_{2w}}p_{x_1x_2\ldots x_r}=p_{i_1i_2\cdots i_{2w}}\ , \tag{35}$$

then we obtain from (34)

$$\sum_{P=1}^{(2w)!}\lambda_P^\sigma p_{\{i_1i_2\cdots i_{2w}\}_P}=0\ . \tag{36}$$

The components of the tensor $p_{x_1x_2\ldots x_{2w}}$ can be selected quite arbitrarily and in spite of it each component on the left-hand side of (36) equals zero. From it

$$\sum_{P=1}^{(2w)!}\lambda_P^\sigma c_P^*\delta_{\{i_1}^{x_1}\delta_{i_2}^{x_2}\cdots\delta_{i_{2w}\}_P}^{x_{2w}}=0,\ \sigma=1,2,\ldots,s\ . \tag{37}$$

The equality (37) gives s independent linear relations between the unknown coefficients c_P^*. If we add (37) to the equations (26), we obtain a system (A) of $2^{4w}+s$ equations in the coefficients c_P^*. The matrix of this system coincides with the matrix of the connected system (27) and (29). Consequently, the rank of the system (A) is $(2w)!$ and one may select from it $(2w)!$ equaitons in such a way that the determinant formed by their system

(B) is non-zero; the system (B) involves all the s equations (37) and $((2w)!-s)$ equations of the system (26) obtained form certain definite values of the indices x_1, x_2, ..., x_{2w}. Solving the system (B) we express the left-hand side of (26) in the form of linear combinations of the right-hand sides of the system (B), i.e. again in the form of the right-hand sides of (26). It means (26) has always a solution. $\Box$

Notes: The solution of (24) is not unique, since we can add to the right-hand side of (26) any linear combination of the left-hand sides of (34).

We supposed two-dimensional space here, but the original proof is valid for arbitrary number of dimensions.

References

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[3] G. B. Gurevich, *Osnovy teorii algebraicheskikh invariantov.* Moskva, The Union of Soviet Socialist Republics: OGIZ, 1937. (in Russian).