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# Discretization of Optimization Problems

# Optimization x E-L equation

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- Optimization problem

$$\min F(u) = \min \int f(x, u, \nabla u) dx$$

- and corresponding E-L equation

$$F'(u) = 0$$

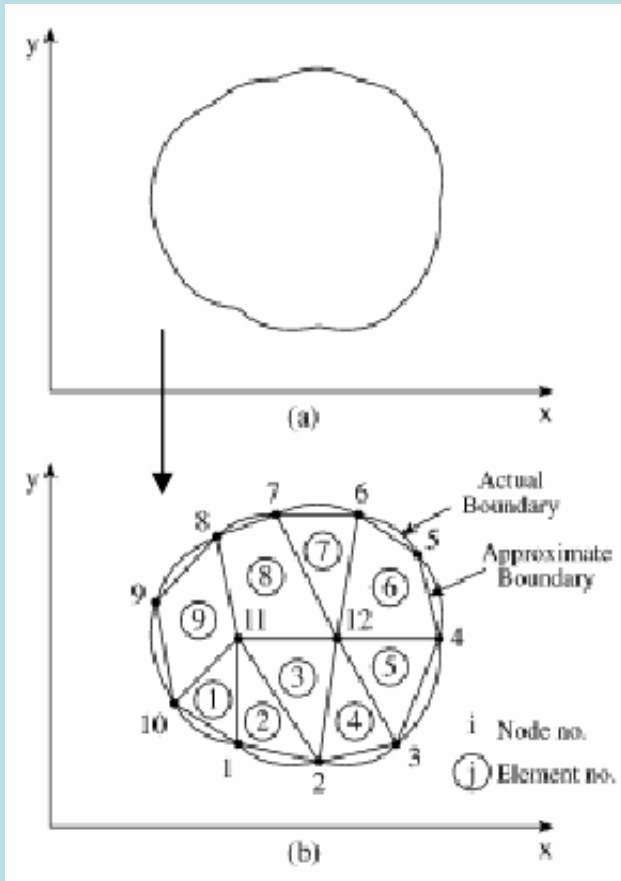
# Discretization

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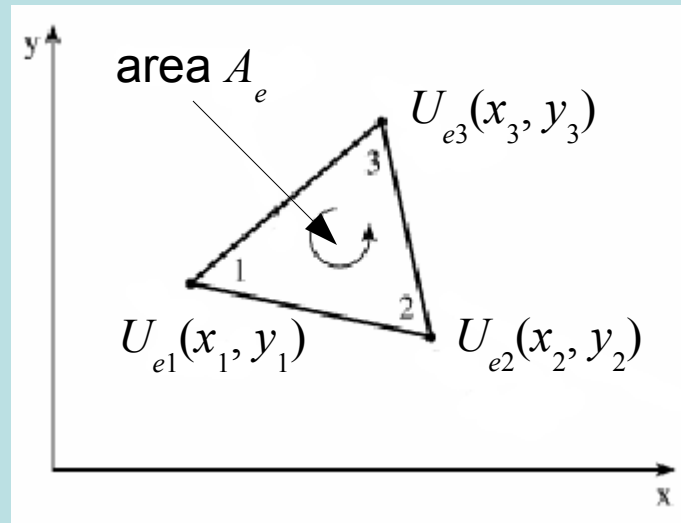
- Finite Element Method (FEM)
  - Solution to integral equation
- Finite Difference Method (FDM)
  - Solution to E-L equation

# Finite Element (FEM)

Partition of the space



Triangular Finite Element



$$u \simeq \sum_{e=1}^N u_e$$

$$u_e(x, y) = a + bx + cy$$

$$\begin{bmatrix} U_{e1} \\ U_{e2} \\ U_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

# FEM

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- Linear approximation of the function inside each element using values in nodes:

$$u_e(x, y) = [1 \quad x \quad y] \frac{1}{2|A_e|} \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{bmatrix} U_{e1} \\ U_{e2} \\ U_{e3} \end{bmatrix}$$

or

$$u_e = \sum_{i=1}^3 \alpha_i(x, y) U_{ei}$$

- Function  $u$  is approximated in the whole element not just in the nodes.

# Solution of Laplace's equation

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- Functional:  $F = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx$

$$f(x, u, \nabla u) = |\nabla u|^2 = u_x^2 + u_y^2$$

- E-L equation:  $-\Delta u = 0$        $\Delta u = u_{xx} + u_{yy}$

- FEM:  $F = \sum_e F_e = \sum_e \frac{1}{2} \int_{A_e} |\nabla u_e|^2 dx$

$$u_e = \sum_{i=1}^3 \alpha_i(x, y) U_{ei}$$


$$\nabla u_e = \sum_{i=1}^3 \nabla \alpha_i(x, y) U_{ei}$$

$$\bullet F = \sum_e F_e = \sum_e \left[ \frac{1}{2} \int_{A_e} |\nabla u_e|^2 dx \right] \quad u_e = \sum_{i=1}^3 \alpha_i(x, y) U_{ei}$$

$$\nabla u_e = \sum_{i=1}^3 \nabla \alpha_i(x, y) U_{ei}$$

$$F_e = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 U_{ei} \left( \int_{A_e} \langle \nabla \alpha_i, \nabla \alpha_j \rangle dx \right) U_{ej}$$

$$F_e = \mathbf{u}_e^T \mathbf{C}^{(e)} \mathbf{u}_e \quad \mathbf{u}_e = \begin{bmatrix} U_{e1} \\ U_{e2} \\ U_{e3} \end{bmatrix} \quad \mathbf{C}^{(e)} = \begin{bmatrix} c_{11}^e & c_{12}^e & c_{13}^e \\ c_{21}^e & c_{22}^e & c_{23}^e \\ c_{31}^e & c_{32}^e & c_{33}^e \end{bmatrix}$$

$$c_{ij}^e = \int_{A_e} \langle \nabla \alpha_i, \nabla \alpha_j \rangle dx$$

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- Leads to a quadratic form:

$$F = \sum_{e=1}^N F_e = \mathbf{u}^T \mathbf{C} \mathbf{u}$$

- $\mathbf{u}$  ... vector of all nodes  $U_{ei}$
- $\mathbf{C}$  ... matrix of all coefficients  $\mathbf{C}^{(e)}$

- Solution:  $\forall i \frac{\partial F}{\partial u_i} = 0 \quad \Rightarrow \quad \mathbf{C} \mathbf{u} = 0$

**A set of linear equations!!!** (Dirichlet B.C.?)



# FEM

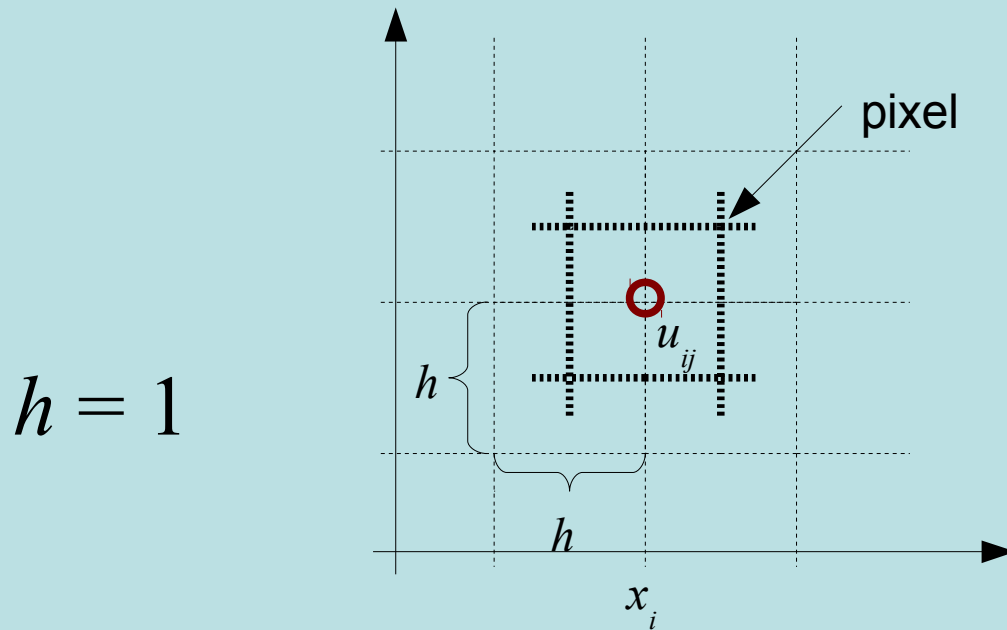
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- Pros
  - Geometrically complex problems
  - Structural mechanics
  
- Cons
  - Often hard to implement

# Finite Difference (FD)

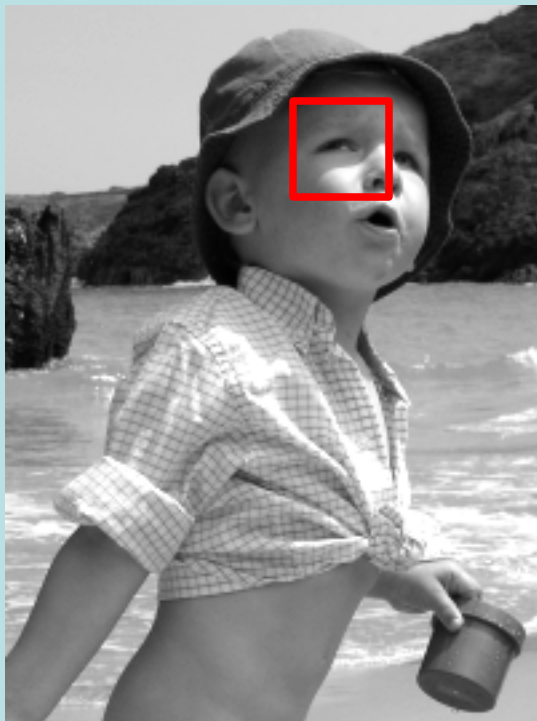
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- Nodes on a uniform grid

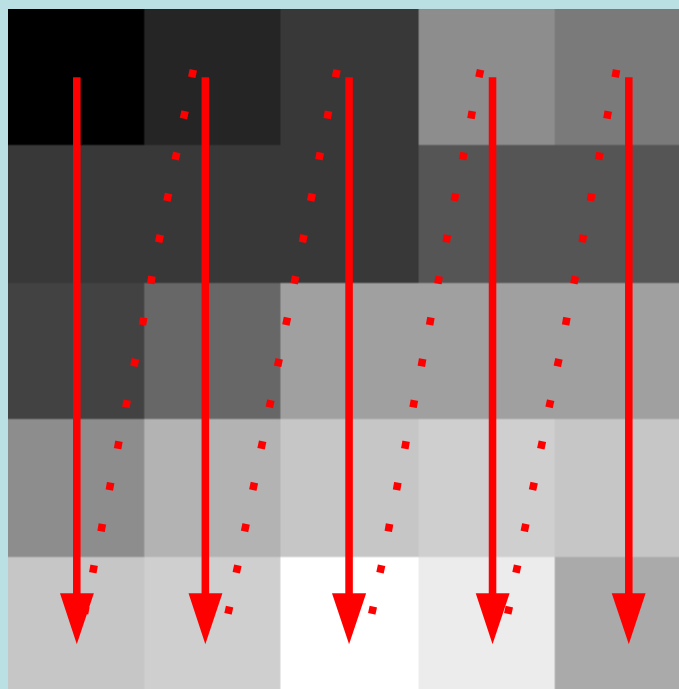


- Approximation of derivative:  $\frac{\partial u}{\partial x}$ ,  $\frac{\partial u}{\partial y}$

# Digital Image



$u(x, y)$



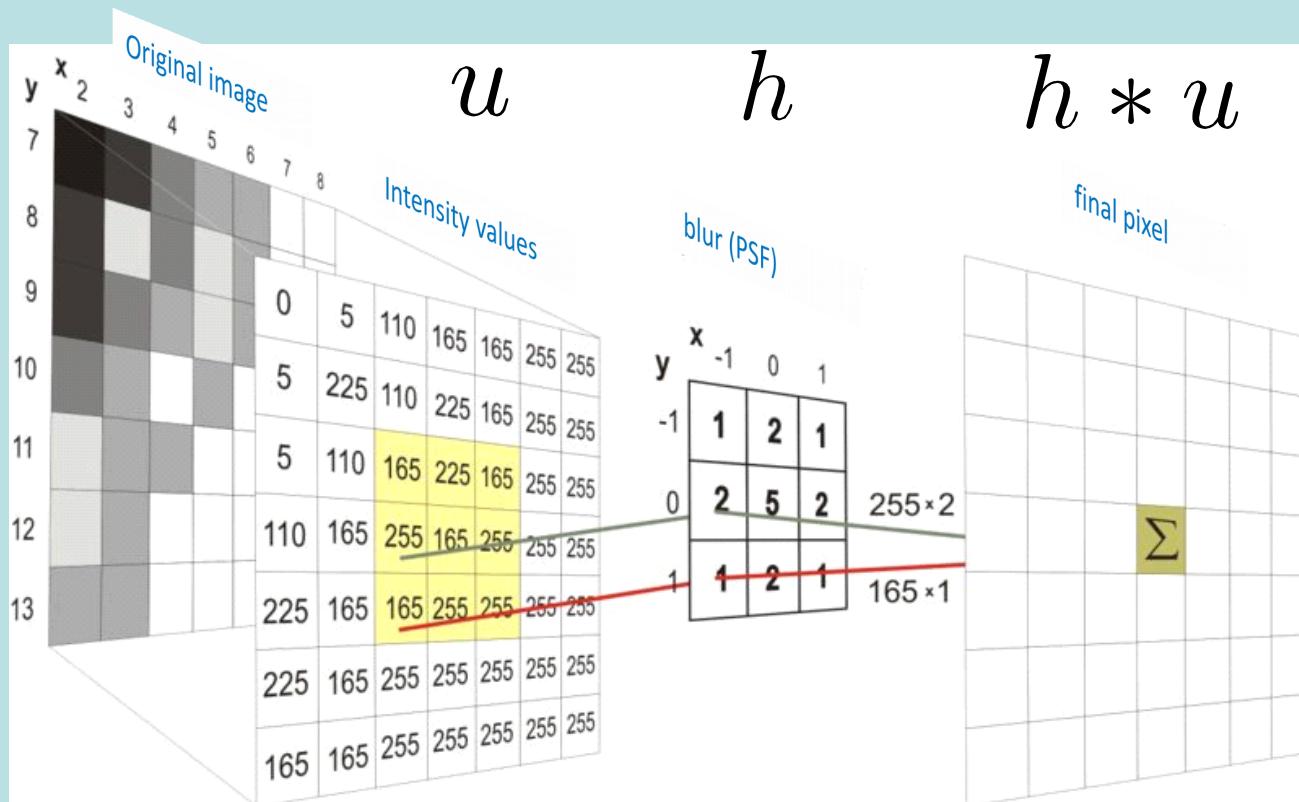
$u_{i,j}$



$\mathbf{u}$

# 2D convolution

$$h * u = \int u(x - s, y - t)h(s, t)dsdt$$



# Convolution

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- Continuous case

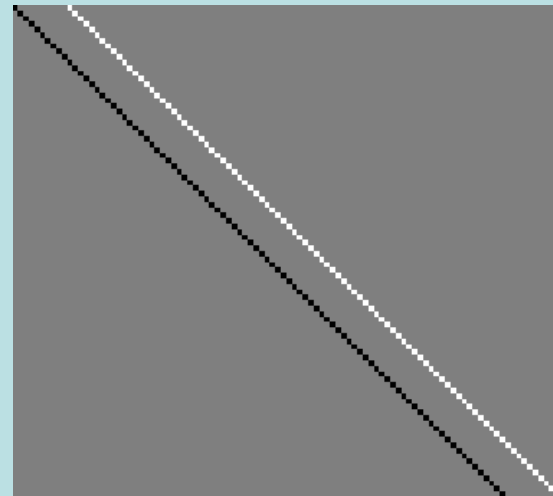
$$h * u = \int h(x - s, y - t)u(s, t)dsdt$$

- Discrete case

$$h * u \approx \mathbf{H}u$$

$$\mathbf{h} = [1, -1]$$

$$\mathbf{H} =$$

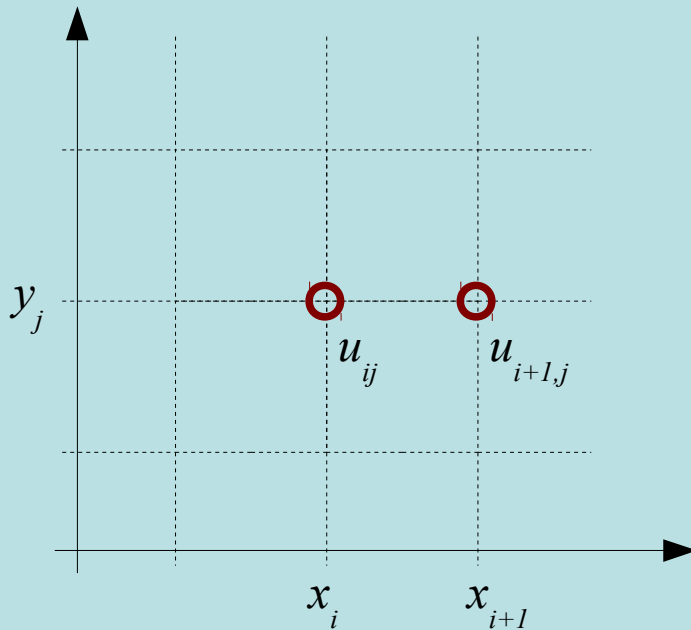


- Boundary effect!!!

# Approximation of Derivatives

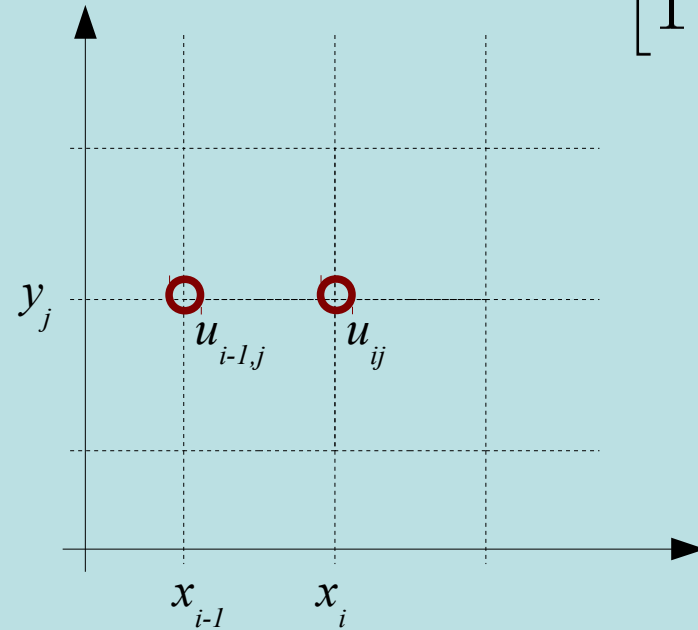
- Based on the Taylor expansion

conv. kernel:  
 $\begin{bmatrix} 1 & -1 \end{bmatrix}$



forward difference

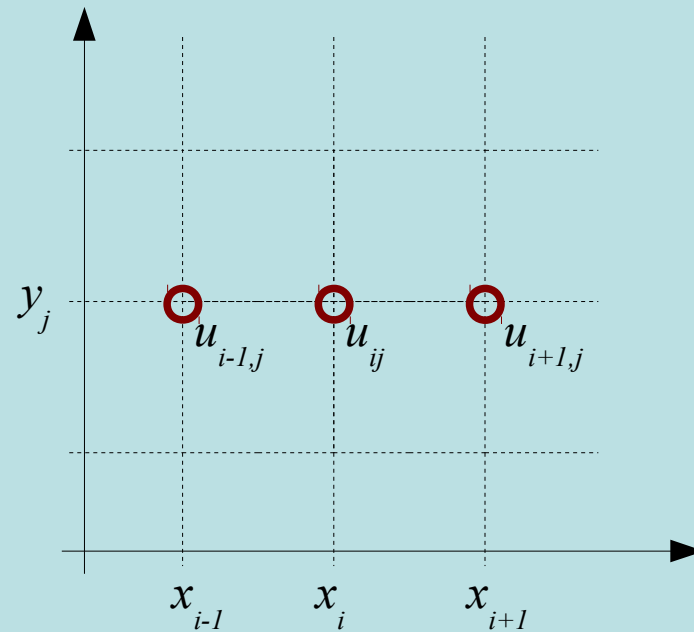
$$\frac{\partial u}{\partial x}(x_i, y_i) \approx \frac{u_{i+1,j} - u_{ij}}{h}$$



backward difference

$$\frac{\partial u}{\partial x}(x_i, y_i) \approx \frac{u_{i,j} - u_{i-1,j}}{h}$$

# Approximation of Derivatives



conv. kernel:

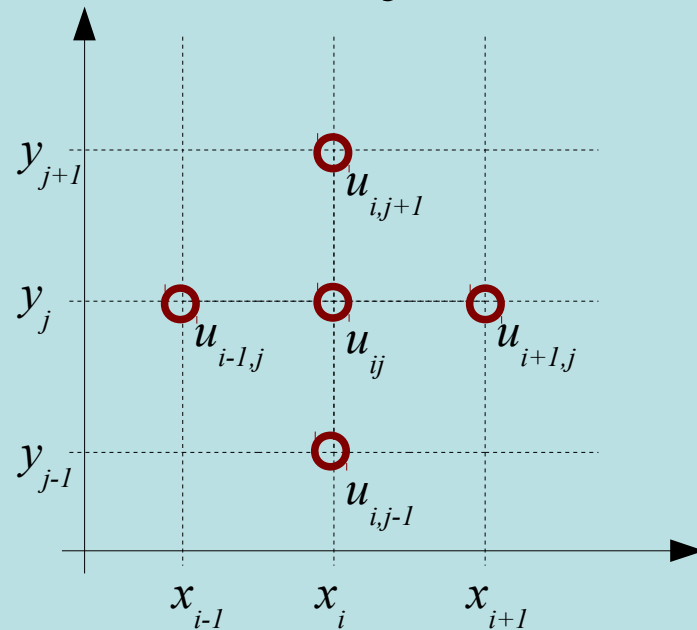
$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

centered difference

$$\frac{\partial u}{\partial x}(x_i, y_i) \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

# Approximation of Laplacian

- Apply forward (backward) differences twice both on  $x$  and  $y$ .



conv. kernel:

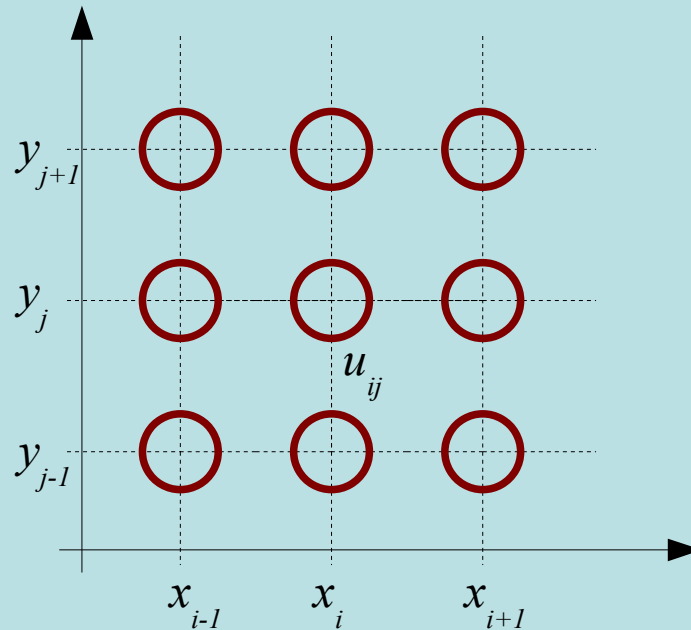
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Delta u(x_i, y_i) \approx \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}$$



# Approximation of Laplacian

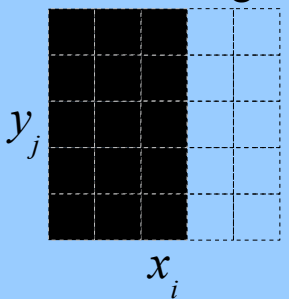
- Rotationally invariant



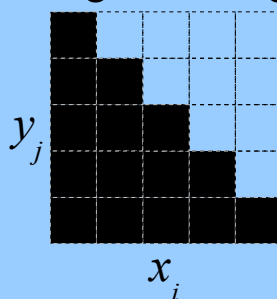
conv. kernel:

$$\frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

vertical edge

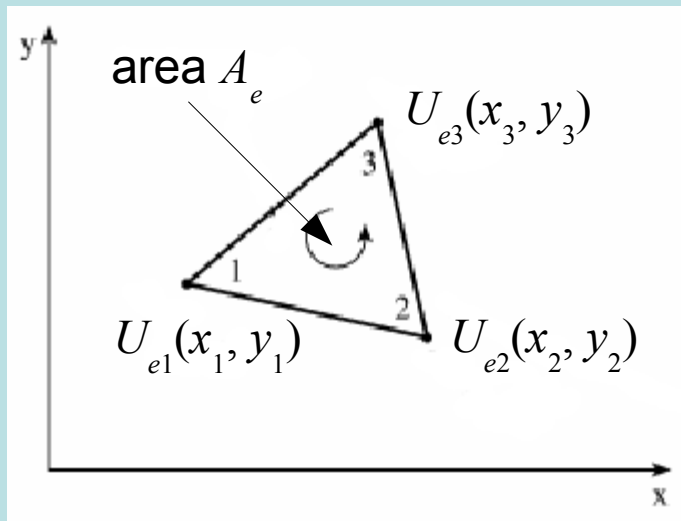


diagonal edge



# Back to Laplace's equation

- FEM:  
complex partitioning

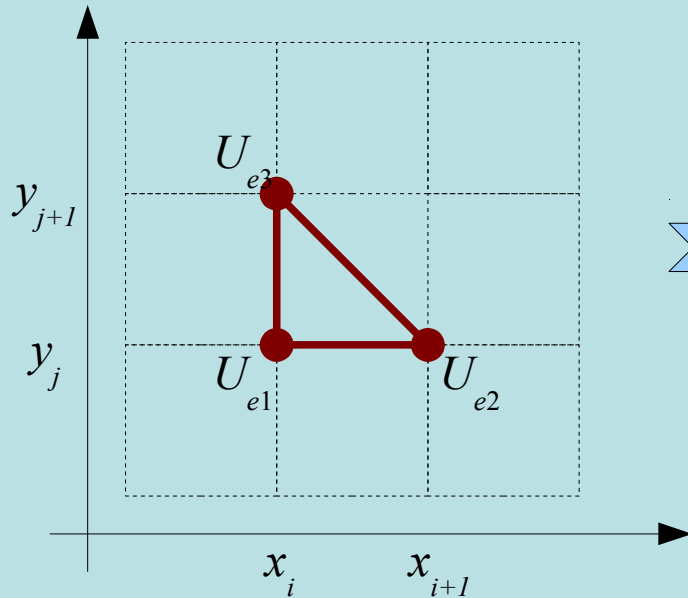


$$F = \sum_{e=1}^N F_e = \mathbf{u}^T \mathbf{C} \mathbf{u}$$

$$c_{ij}^e = \int_{A_e} \langle \nabla \alpha_i, \nabla \alpha_j \rangle dx$$

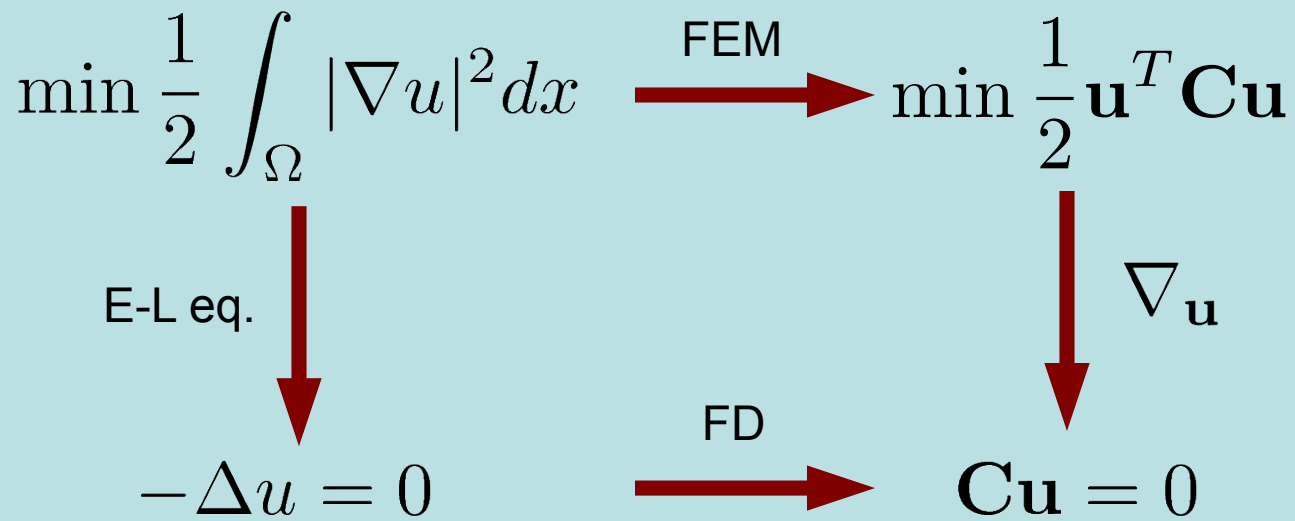
...

- Regular grid & lin.approx. of  $u$  on  $A$



$C$  is a discrete  
- Laplacian.  
convolution with

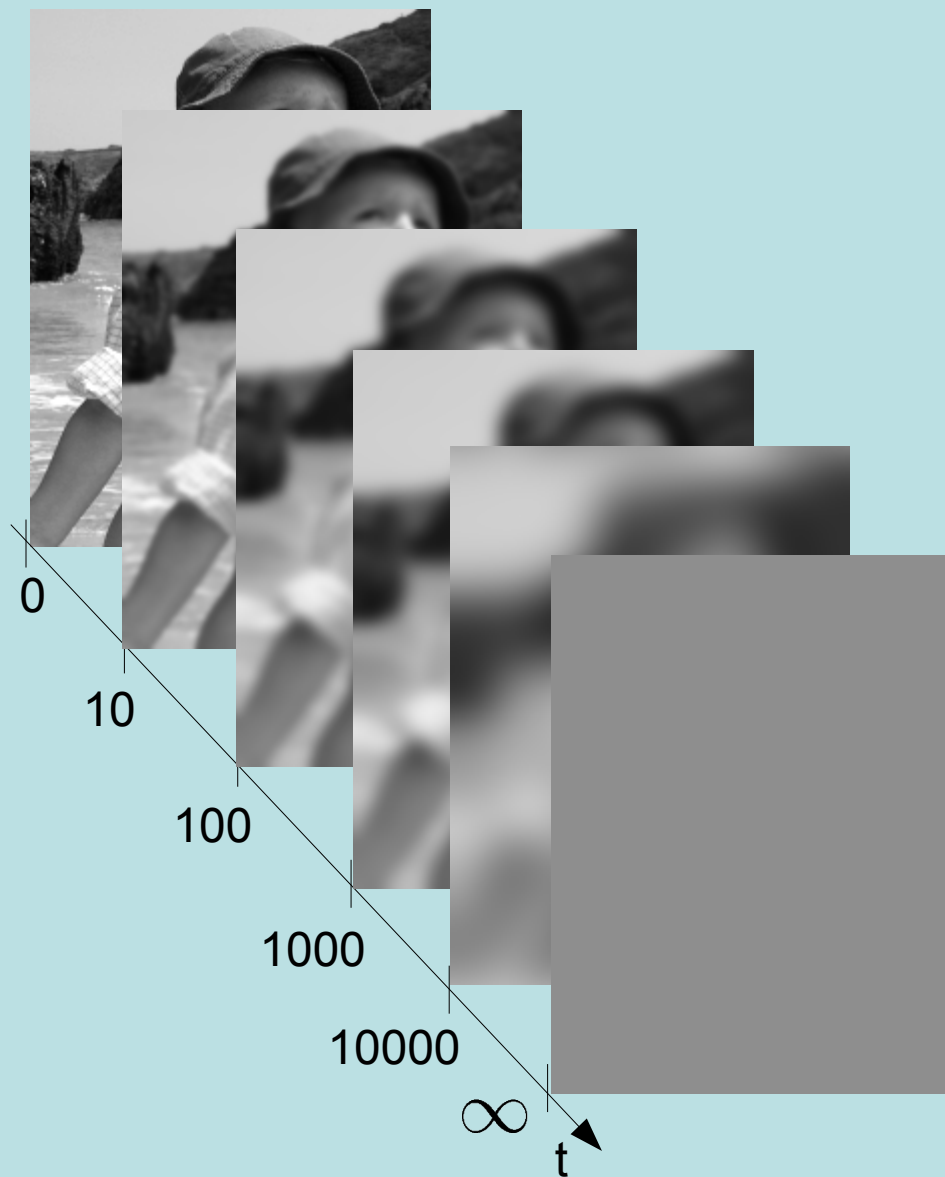
$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$



# Evolution of Laplace's Equation

$$u_t = \Delta u$$

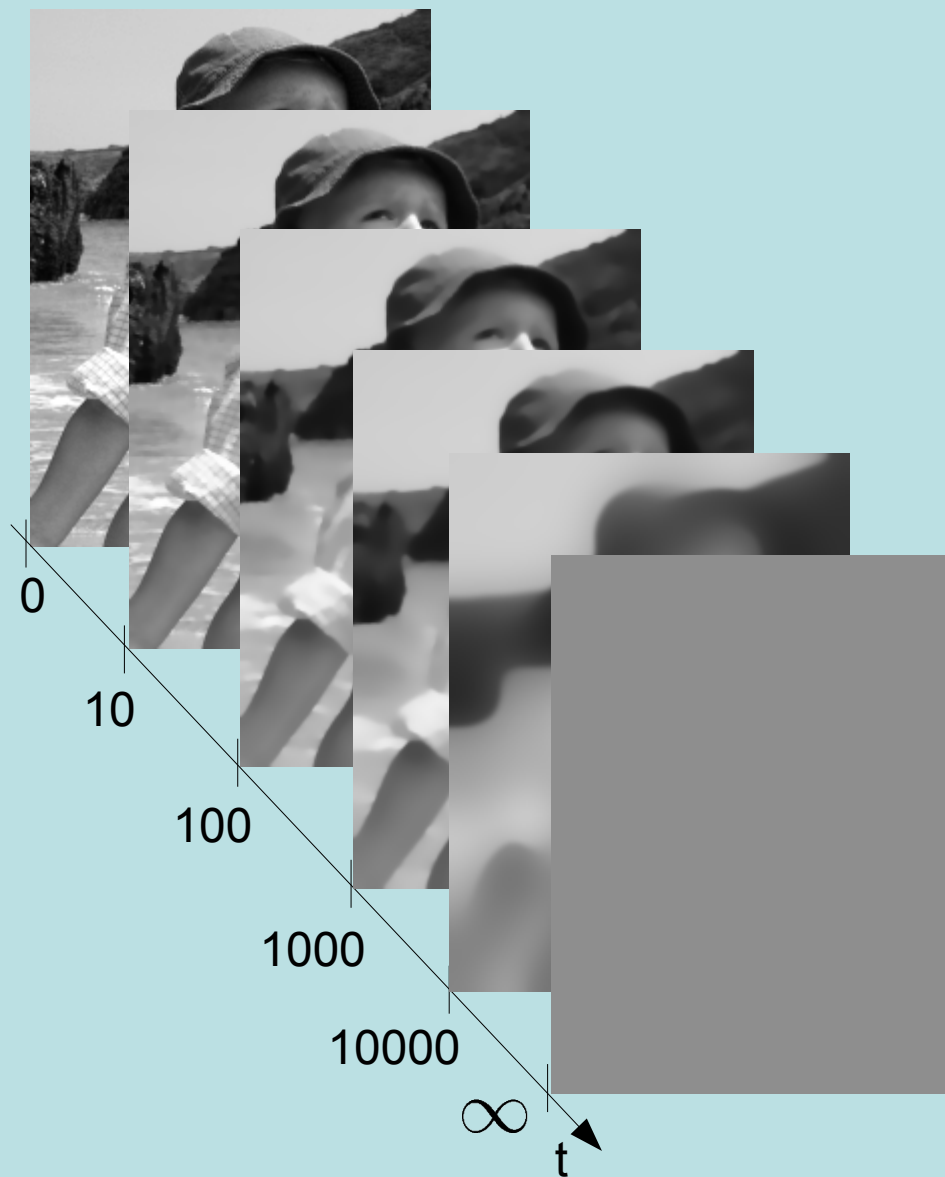
$$\mathbf{u}_{k+1} = \mathbf{u}_k - \alpha \mathbf{C} \mathbf{u}_k$$



# Evolution of TV Equation

$$u_t = \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right)$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k - \alpha \mathbf{L}_{\nabla \mathbf{u}_k} \mathbf{u}_k$$



# Isotropic & Anisotropic Diffusion

$$\min \int |\nabla u|^2$$



$$\min \int |\nabla u|$$

