

Handling Blur

Image Restoration

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Invariants to convolution



$$g(x, y) = (f * h)(x, y)$$



$$I(f * h) = I(f)$$

for any admissible h

$$I(f) \equiv \frac{\mathcal{F}(f)}{\mathcal{F}(Pf)}$$

1990s: Flusser et al., centrosymmetry assumption

Projection operators

$$I(f) \equiv \frac{\mathcal{F}(f)}{\mathcal{F}(Pf)}$$

\mathcal{I} – the image space

$\mathcal{S} \subset \mathcal{I}$ – the PSF space closed under convolution

P : projection operator $\mathcal{I} \rightarrow \mathcal{S}$, $P^2 = P$

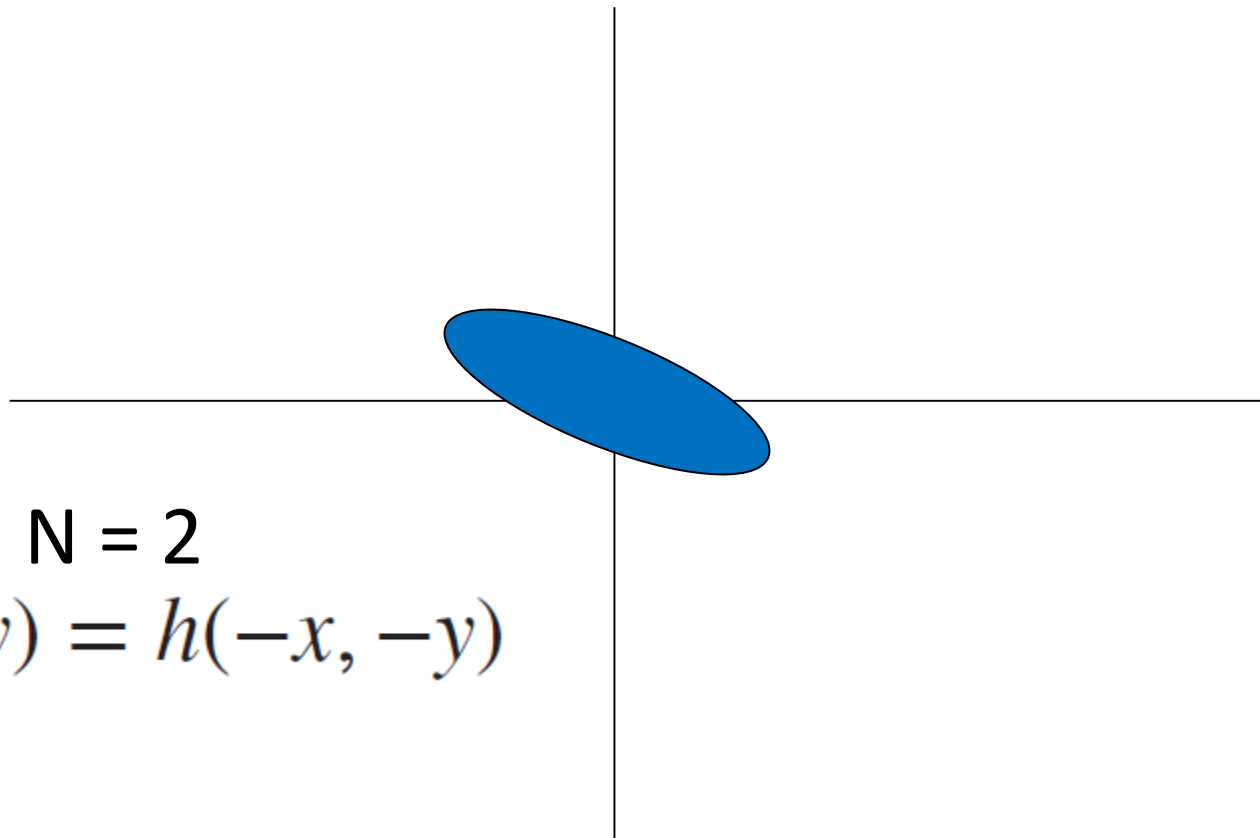
Applicability of the FTBI

- Which PSF spaces \mathcal{S} are closed under convolution?
- For which \mathcal{S} can we construct \mathcal{P} ?
- Are there any $[\mathcal{S}, \mathcal{P}]$ with non-trivial FTBI invariants?
- If yes, do they correspond with real-life PSF's?
- Calculation of the invariants in the image domain?

The answer is ...

Assumptions on the PSF

The PSF has N-fold rotation symmetry

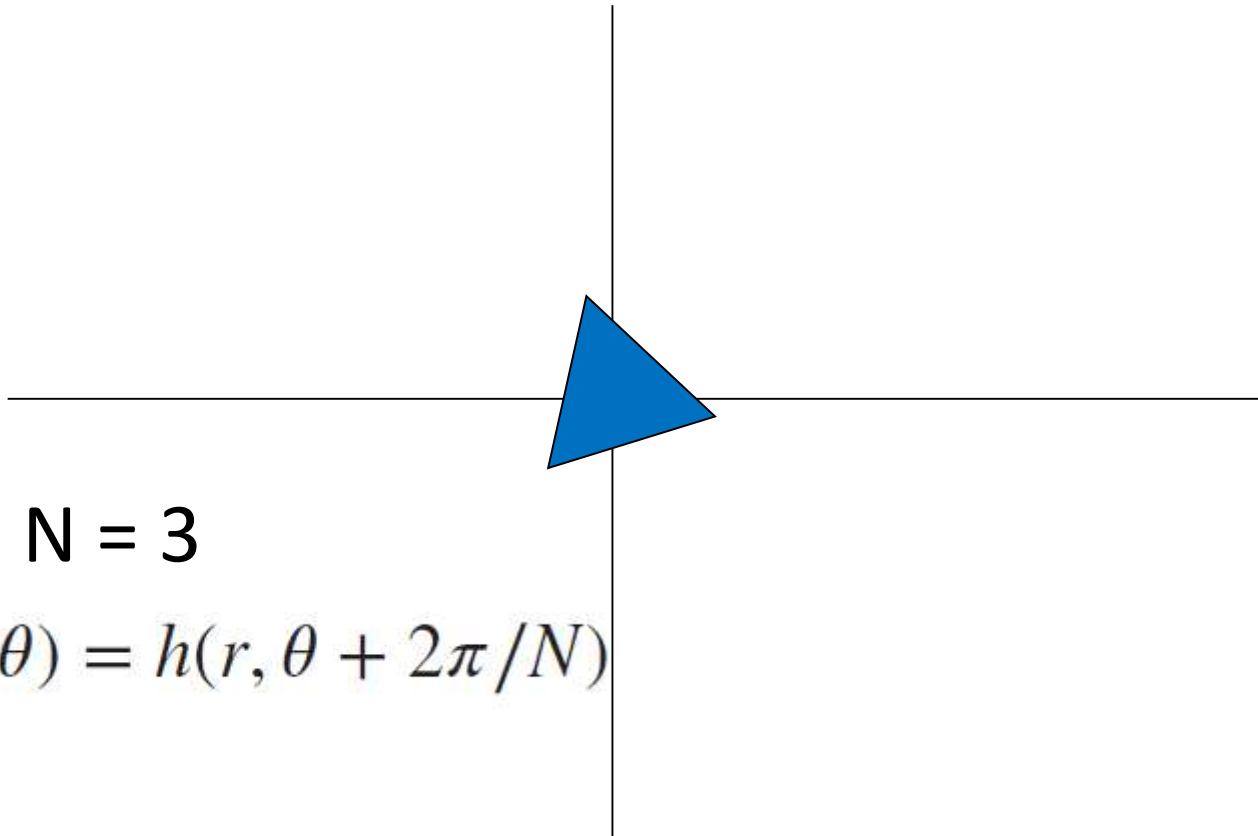


$$N = 2$$

$$h(x, y) = h(-x, -y)$$

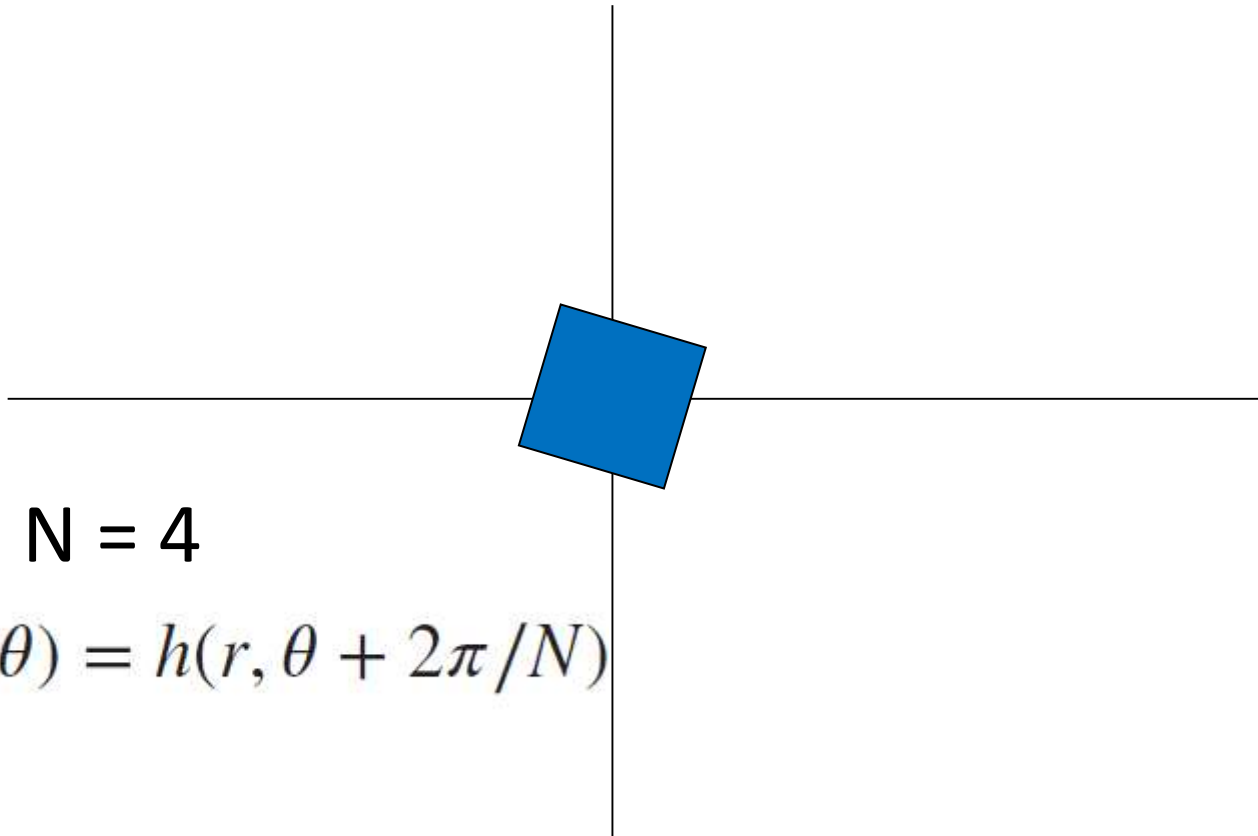
Assumptions on the PSF

The PSF has N -fold rotation symmetry



Assumptions on the PSF

The PSF has N-fold rotation symmetry

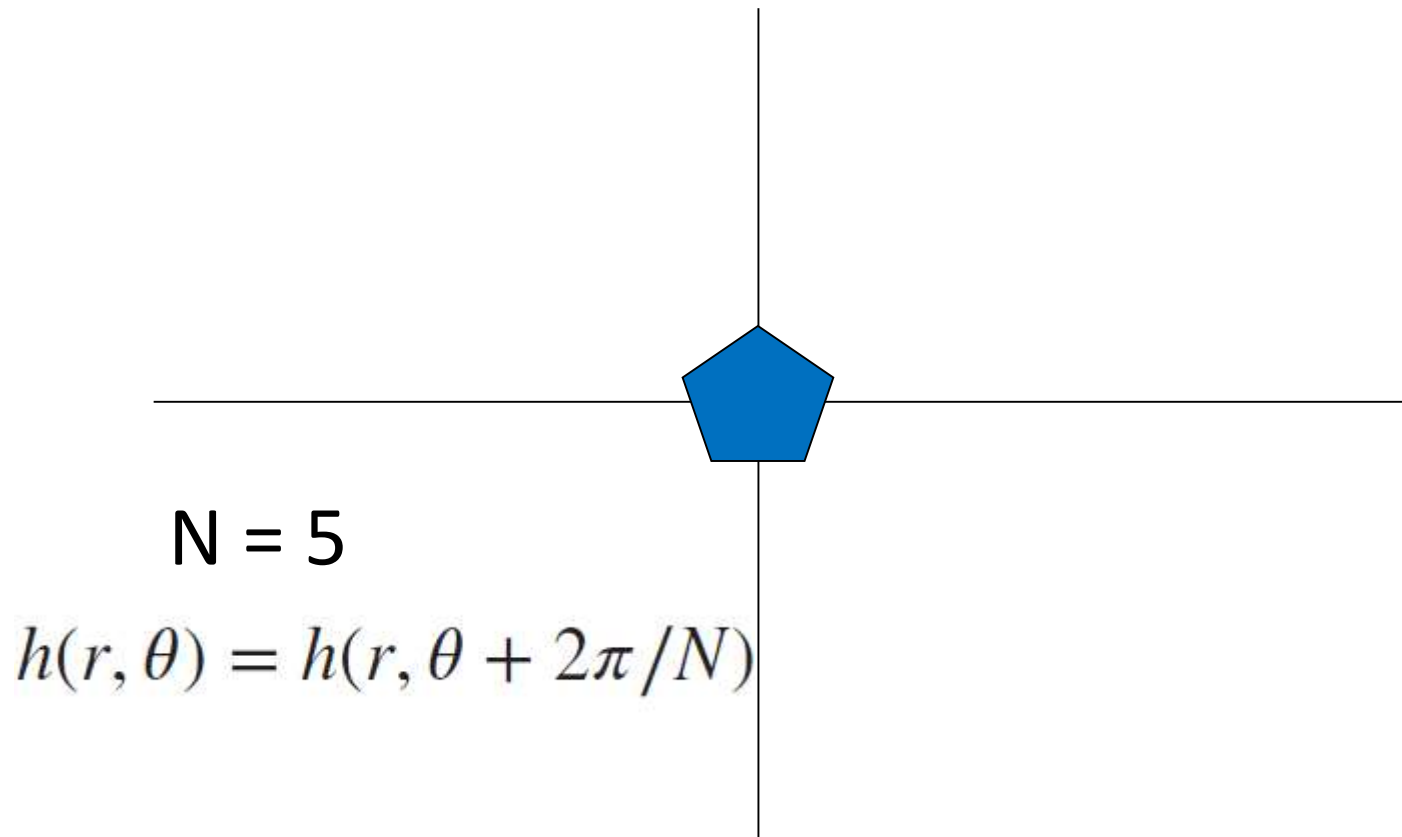


$$N = 4$$

$$h(r, \theta) = h(r, \theta + 2\pi/N)$$

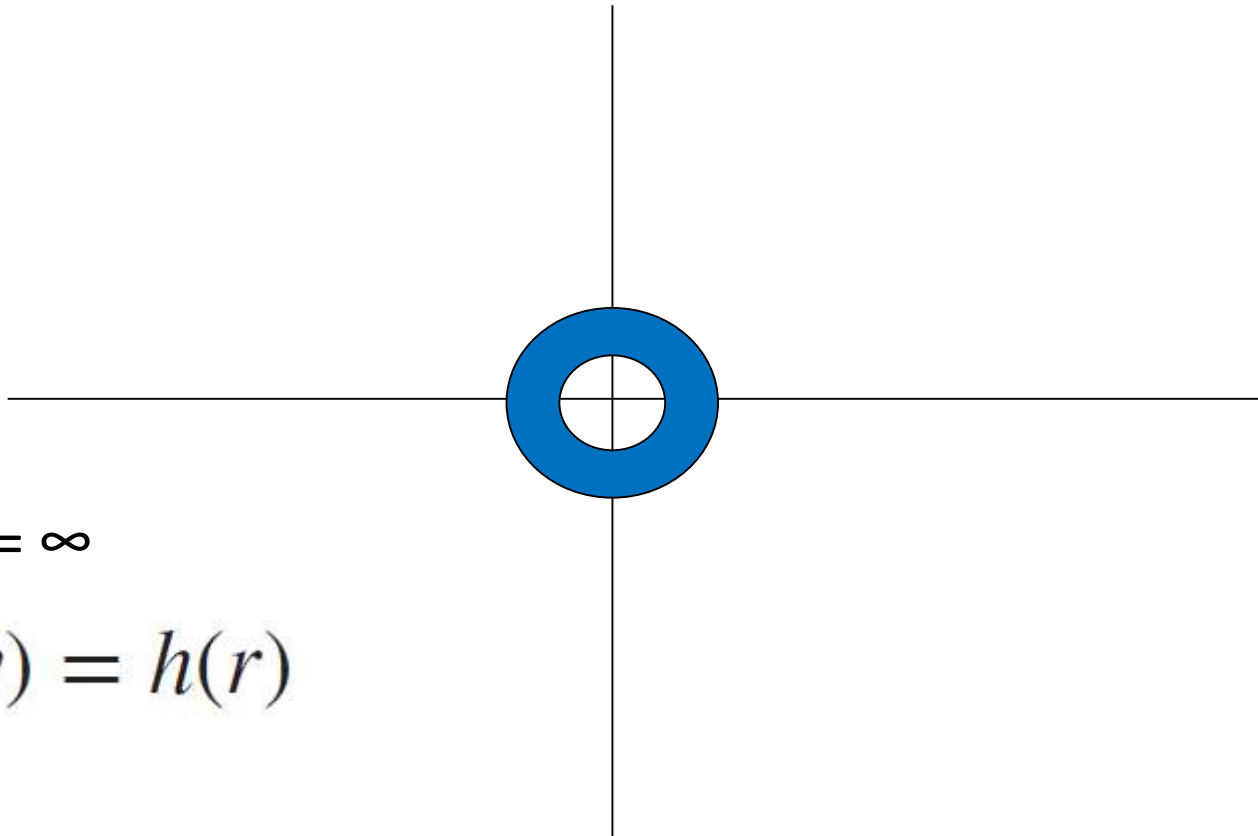
Assumptions on the PSF

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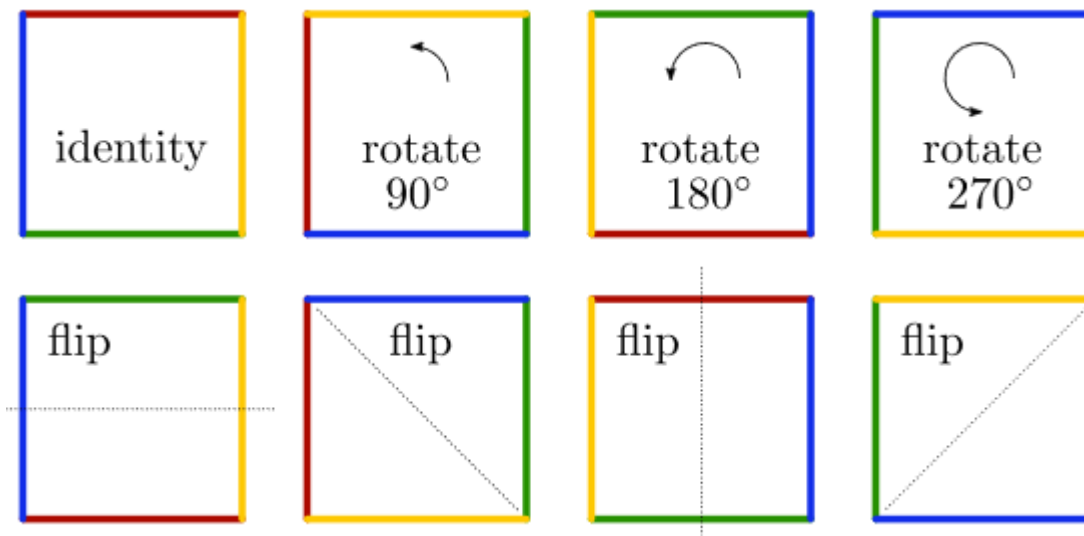


$$N = \infty$$

$$h(x, y) = h(r)$$

Assumptions on the PSF

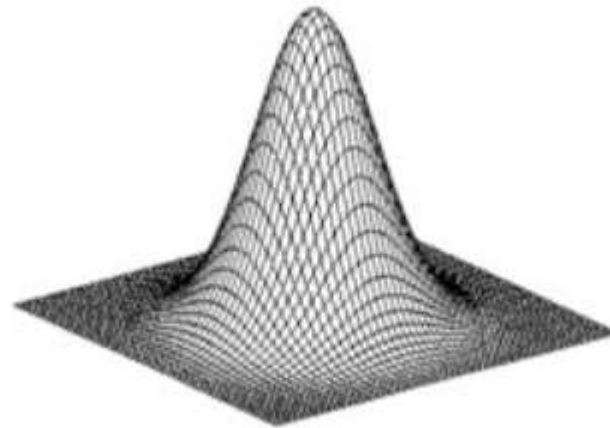
The PSF has N-fold dihedral symmetry



N-fold rotation symmetry and N mirror reflections

Assumptions on the PSF

The PSF has a Gaussian shape

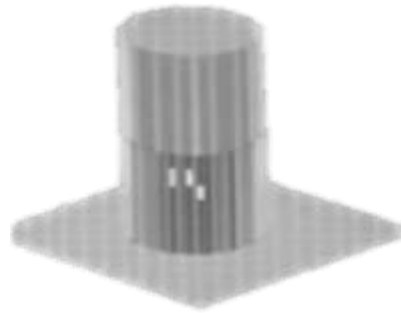


$$G(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Assumptions on the PSF

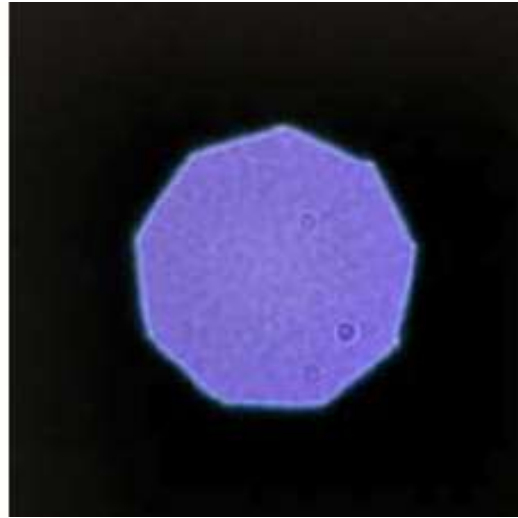
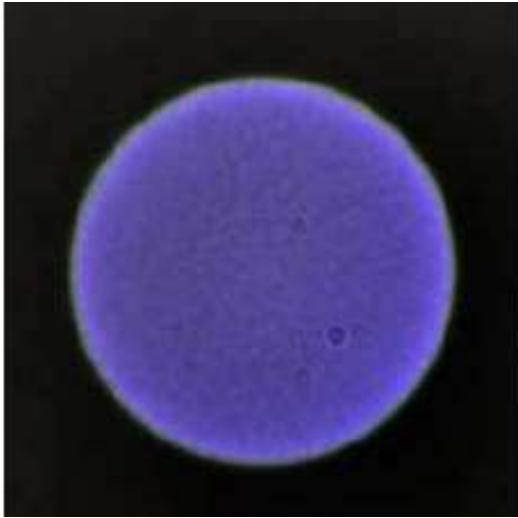
Out-of-focus blur, "geometric optic approximation"

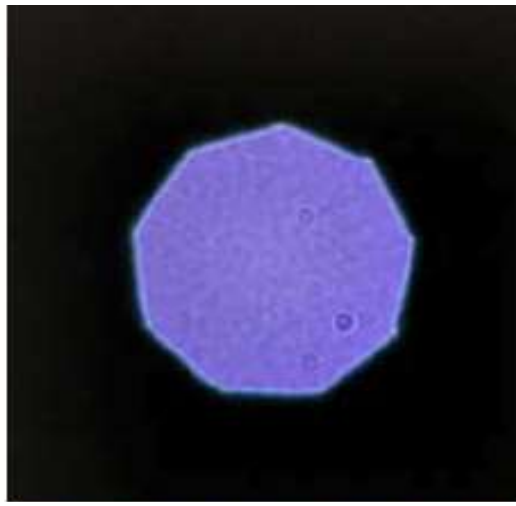
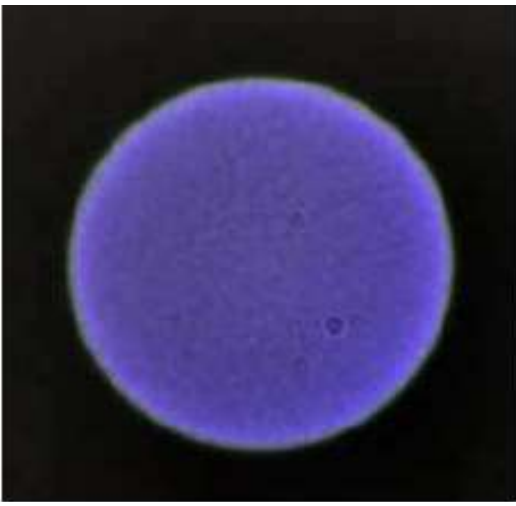
$$g(x, y) = (f * h)(x, y)$$



Is it realistic?

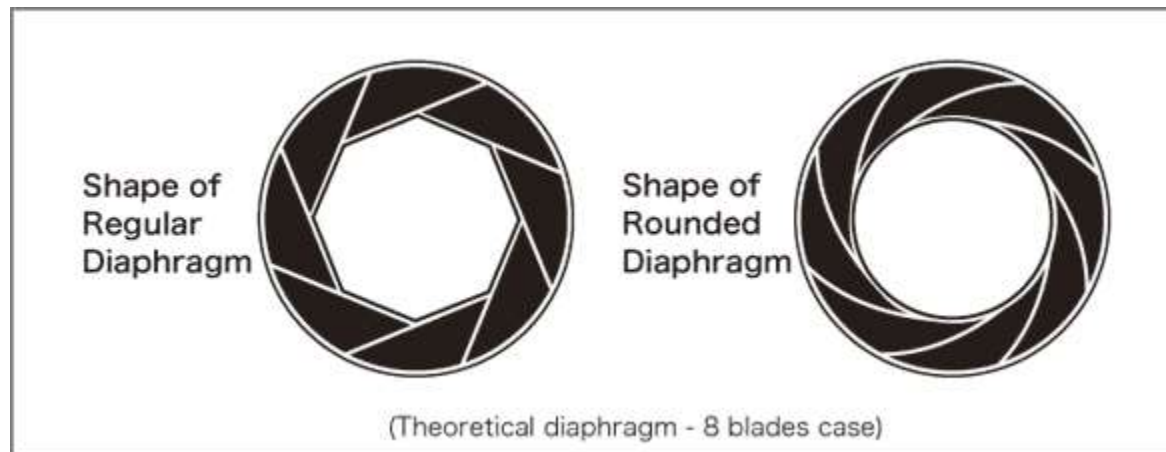
Assumptions on the PSF





Handling Blur - ICPR 2016

Assumptions on the PSF - apertures



Assumptions on the PSF - bokeh

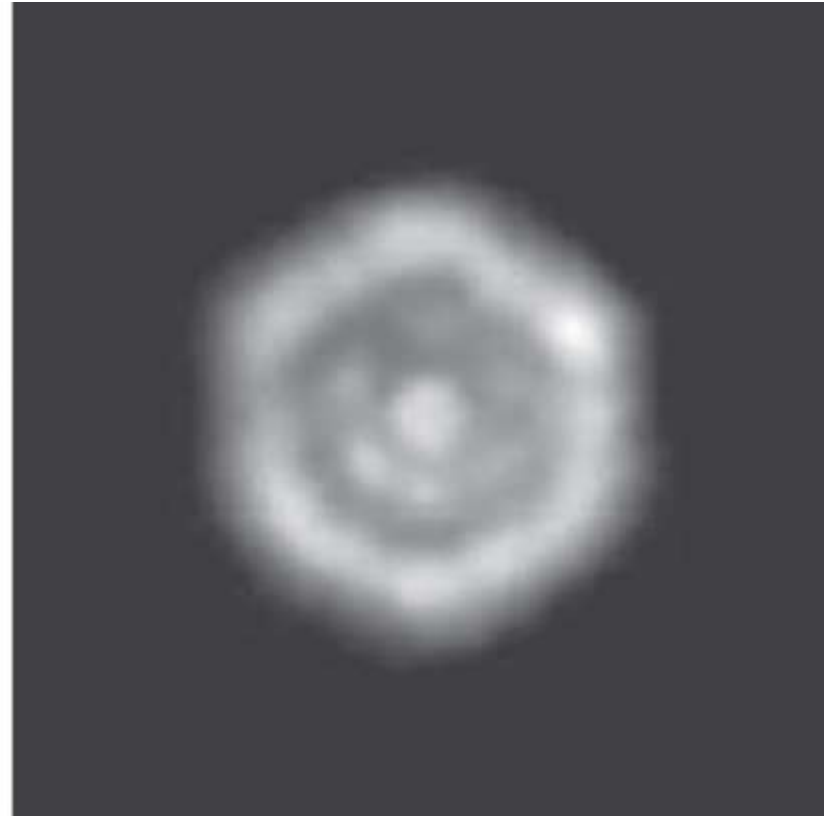
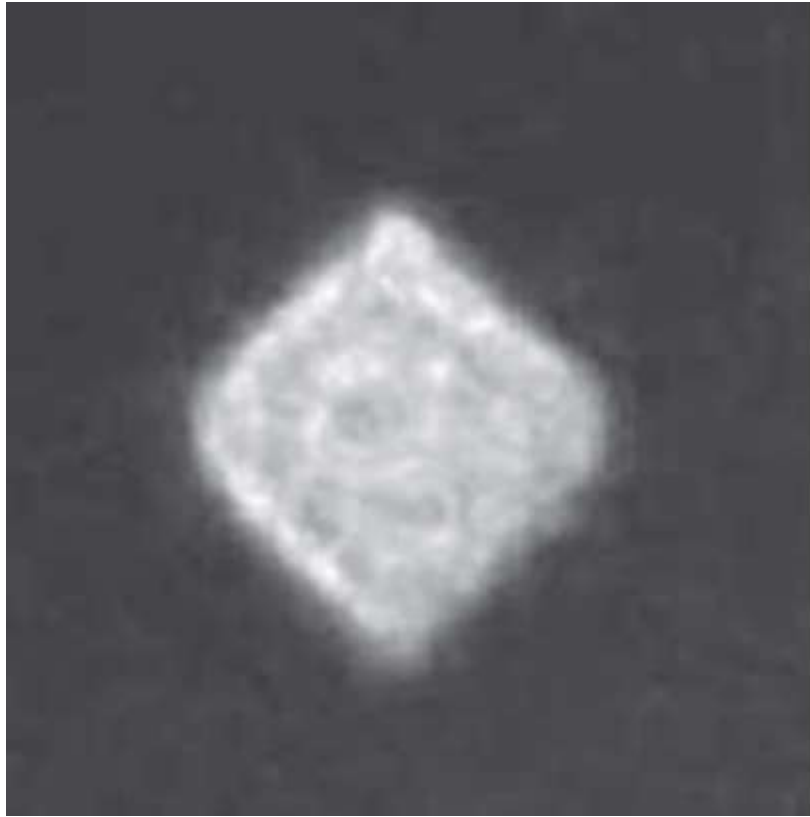


∞ -fold rotation symmetry

Assumptions on the PSF - bokeh



Assumptions on the PSF - dihedral



What are moments?

Moments are “projections” of the image function into a polynomial basis

$f(x, y)$ – piecewise continuous image function defined on bounded $\Omega \subset \mathcal{R} \times \mathcal{R}$

$\{\mathcal{P}_{pq}(x, y)\}$ – set of polynomials defined on Ω

$$M_{pq} = \iint \mathcal{P}_{pq}(x, y) f(x, y) dx dy$$

The most common moments

Geometric moments

$$m_{pq}^{(f)} = \int_{R^2} x^p y^q f(x, y) dx dy$$

(p + q) - the order of the moment

0th order - area

1st order - center of gravity $x_t = \frac{m_{10}}{m_{00}}, \quad y_t = \frac{m_{01}}{m_{00}}$

2nd order - moments of inertia

3rd order - skewness

The most common moments

Central moments

$$\mu_{pq}^{(f)} = \int_{R^2} (x - x_t)^p (y - y_t)^q f(x, y) dx dy$$

$$x_t = m_{10}/m_{00}, \quad y_t = m_{01}/m_{00}$$



The most common moments

Normalized central moments

$$\nu_{pq} = \frac{\mu_{pq}}{\mu_{00}^w} \quad w = \frac{p + q}{2} + 1$$



The most common moments

Complex moments

$$c_{pq}^{(f)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + iy)^p (x - iy)^q f(x, y) dx dy$$

The moments under convolution

$$g(x, y) = (f * h)(x, y)$$

Geometric/central

$$\mu_{pq}^{(g)} = \sum_{k=0}^p \sum_{j=0}^q \binom{p}{k} \binom{q}{j} \mu_{kj}^{(h)} \mu_{p-k, q-j}^{(f)}$$

Complex

$$c_{pq}^{(g)} = \sum_{k=0}^p \sum_{j=0}^q \binom{p}{k} \binom{q}{j} c_{kj}^{(h)} c_{p-k, q-j}^{(f)}$$

Applicability of the FTBI

- Which PSF spaces \mathcal{S} are closed under convolution?
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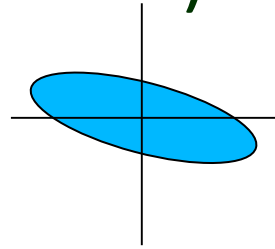
The answer is ...

Intuition: How to eliminate $c_{kj}^{(h)}$?

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- centrosymmetric PSF ($N = 2$) with a unit integral

$$h(x, y) = h(-x, -y)$$



90% of blur invariants two fold rotation symmetry ...

Intuition: How to eliminate $c_{kj}^{(h)}$?

$$c_{pq}^{(g)} = \sum_{k=0}^p \sum_{j=0}^q \binom{p}{k} \binom{q}{j} c_{kj}^{(h)} c_{p-k, q-j}^{(f)}$$

even

energy preserving

$$c_{00}^{(g)} = c_{00}^{(f)} c_{00}^{(h)} = c_{00}^{(f)}$$

$$c_{10}^{(g)} = c_{10}^{(f)} c_{00}^{(h)} + c_{00}^{(f)} c_{10}^{(h)} = c_{10}^{(f)} + c_{00}^{(f)} c_{10}^{(h)}$$

$$c_{20}^{(g)} = c_{20}^{(f)} + 2c_{10}^{(f)} c_{10}^{(h)} + c_{00}^{(f)} c_{20}^{(h)}$$

$$c_{30}^{(g)} = c_{30}^{(f)} + 3c_{20}^{(f)} c_{10}^{(h)} + 3c_{10}^{(f)} c_{20}^{(h)} + c_{00}^{(f)} c_{30}^{(h)}$$

Invariants to centrosymmetric convolution

$$C(3, 0) = \mu_{30},$$

$$C(2, 1) = \mu_{21},$$

$$C(1, 2) = \mu_{12},$$

$$C(0, 3) = \mu_{03}.$$

Invariants to centrosymmetric convolution

$$C(5,0) = \mu_{50} - \frac{10\mu_{30}\mu_{20}}{\mu_{00}},$$

$$C(4,1) = \mu_{41} - \frac{2}{\mu_{00}}(3\mu_{21}\mu_{20} + 2\mu_{30}\mu_{11}),$$

$$C(3,2) = \mu_{32} - \frac{1}{\mu_{00}}(3\mu_{12}\mu_{20} + \mu_{30}\mu_{02} + 6\mu_{21}\mu_{11}),$$

$$C(2,3) = \mu_{23} - \frac{1}{\mu_{00}}(3\mu_{21}\mu_{02} + \mu_{03}\mu_{20} + 6\mu_{12}\mu_{11}),$$

$$C(1,4) = \mu_{14} - \frac{2}{\mu_{00}}(3\mu_{12}\mu_{02} + 2\mu_{03}\mu_{11}),$$

$$C(0,5) = \mu_{05} - \frac{10\mu_{03}\mu_{02}}{\mu_{00}}.$$

Invariants to centrosymmetric convolution

$$C(p, q)^{(f)} = \mu_{pq}^{(f)} - \frac{1}{\mu_{00}^{(f)}} \sum_{\substack{n=0 \\ 0 < n+m < p+q}}^p \sum_{m=0}^q \binom{p}{n} \binom{q}{m} C(p-n, q-m)^{(f)} \cdot \mu_{nm}^{(f)}$$

$$K(p, q)^{(f)} = c_{pq}^{(f)} - \frac{1}{c_{00}^{(f)}} \sum_{\substack{n=0 \\ 0 < n+m < p+q}}^p \sum_{m=0}^q \binom{p}{n} \binom{q}{m} K(p-n, q-m)^{(f)} \cdot c_{nm}^{(f)}$$

where $(p + q)$ is odd

What is the intuitive meaning of the invariants?
“measure of anti-symmetry”

What about FT domain?

Convolution invariants in FT domain

$$g = f * h$$

$$G = F \cdot H$$

$$|G| = |F| \cdot |H|$$

$$\text{ph}G = \text{ph}F + \text{ph}H$$

Convolution invariants in FT domain

Centrosymmetric $h(x, y) \implies$ real $H(u, v)$

$$\text{ph}H \in \{0; \pi\}$$

$$\tan(\text{ph}G) = \tan(\text{ph}F)$$

Back to projection operators

Invariants to convolution

$$I(f) \equiv \frac{\mathcal{F}(f)}{\mathcal{F}(Pf)}$$

\mathcal{I} – the image space

$\mathcal{S} \subset \mathcal{I}$ – the PSF space closed under convolution

P : projection operator $\mathcal{I} \rightarrow \mathcal{S}$, $P^2 = P$

Invariants to convolution

$$\mathcal{F}(Pf)(\mathbf{u}) \cdot I(f)(\mathbf{u}) = \mathcal{F}(f)(\mathbf{u})$$

Taylor expansion
of Fourier transform

$$\mathcal{F}(f)(\mathbf{u}) = \sum_{\mathbf{p}} \frac{(-2\pi i)^{|\mathbf{p}|}}{\mathbf{p}!} m_{\mathbf{p}}^{(f)} \mathbf{u}^{\mathbf{p}}$$

$$\sum_{\mathbf{p} \in D} \frac{(-2\pi i)^{|\mathbf{p}|}}{\mathbf{p}!} m_{\mathbf{p}}^{(f)} \mathbf{u}^{\mathbf{p}} \cdot \sum_{\mathbf{p}} \frac{(-2\pi i)^{|\mathbf{p}|}}{\mathbf{p}!} A_{\mathbf{p}} \mathbf{u}^{\mathbf{p}} = \sum_{\mathbf{p}} \frac{(-2\pi i)^{|\mathbf{p}|}}{\mathbf{p}!} m_{\mathbf{p}}^{(f)} \mathbf{u}^{\mathbf{p}}$$

Invariants to convolution

$$\sum_{\mathbf{p} \in D} \frac{(-2\pi i)^{|\mathbf{p}|}}{\mathbf{p}!} m_{\mathbf{p}}^{(f)} \mathbf{u}^{\mathbf{p}} \cdot \sum_{\mathbf{p}} \frac{(-2\pi i)^{|\mathbf{p}|}}{\mathbf{p}!} A_{\mathbf{p}} \mathbf{u}^{\mathbf{p}} = \sum_{\mathbf{p}} \frac{(-2\pi i)^{|\mathbf{p}|}}{\mathbf{p}!} m_{\mathbf{p}}^{(f)} \mathbf{u}^{\mathbf{p}}$$

comparing the coefficients of the same powers of \mathbf{u}

$$\sum_{\mathbf{k} \in D} \frac{(-2\pi i)^{|\mathbf{k}|}}{\mathbf{k}!} \frac{(-2\pi i)^{|\mathbf{p}-\mathbf{k}|}}{(\mathbf{p}-\mathbf{k})!} m_{\mathbf{k}} A_{\mathbf{p}-\mathbf{k}} = \frac{(-2\pi i)^{|\mathbf{p}|}}{\mathbf{p}!} m_{\mathbf{p}}$$

$$\sum_{\mathbf{k} \in D} \binom{\mathbf{p}}{\mathbf{k}} m_{\mathbf{k}} A_{\mathbf{p}-\mathbf{k}} = m_{\mathbf{p}}$$

General definition of blur invariants in the image domain

$$\sum_{\mathbf{k} \in D} \binom{\mathbf{p}}{\mathbf{k}} m_{\mathbf{k}} A_{\mathbf{p}-\mathbf{k}} = m_{\mathbf{p}}$$

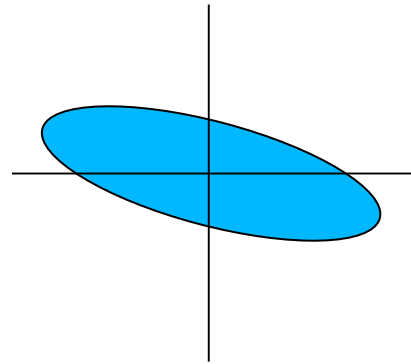
$$m_{\mathbf{0}} A_{\mathbf{p}} = m_{\mathbf{p}} - \sum_{\substack{\mathbf{k} \in D \\ \mathbf{k} \neq \mathbf{0}}} \binom{\mathbf{p}}{\mathbf{k}} m_{\mathbf{k}} A_{\mathbf{p}-\mathbf{k}}$$

$A_{\mathbf{p}} \sim$ moments of the primordial image $I(f)$ of f

No need of $I(f)$ construction ! No need of FT !

Invariants to centrosymmetric PSF

centrosymmetric PSF ($N = 2$) with a unit integral



$$\mathcal{S} \equiv \mathcal{C}_2 = \{h \in \mathcal{I} \mid h(x, y) = h(-x, -y)\}$$

Invariants to centrosymmetric PSF



Invariants to centrosymmetric PSF

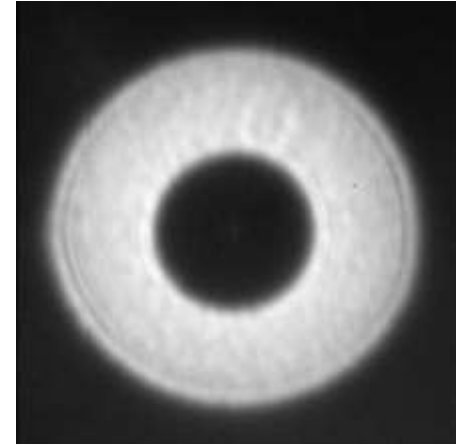
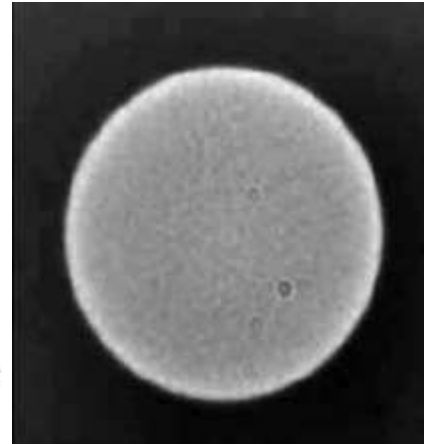
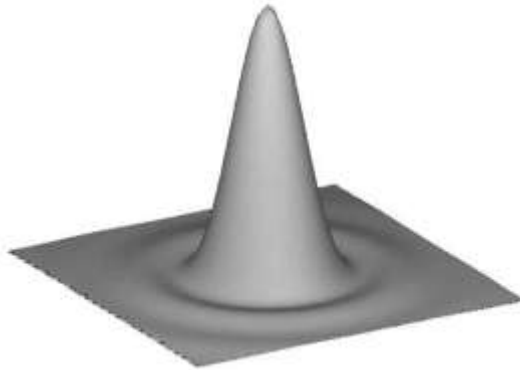
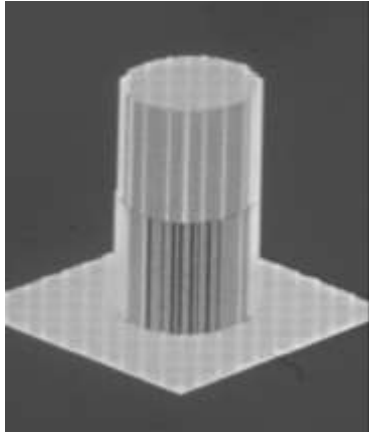
$$(P_2 f)(x, y) = (f(x, y) + f(-x, -y)) / 2$$

$$I_2(f) = \frac{F}{P_2 F} \quad I_2(f) = 1 + i \tan(\text{ph } F)$$

$$K(p, q)^{(f)} = c_{pq}^{(f)} - \frac{1}{c_{00}^{(f)}} \sum_{n=0}^p \sum_{\substack{m=0 \\ 0 < n+m < p+q}}^q \binom{p}{n} \binom{q}{m} K(p-n, q-m)^{(f)} \cdot c_{nm}^{(f)}$$

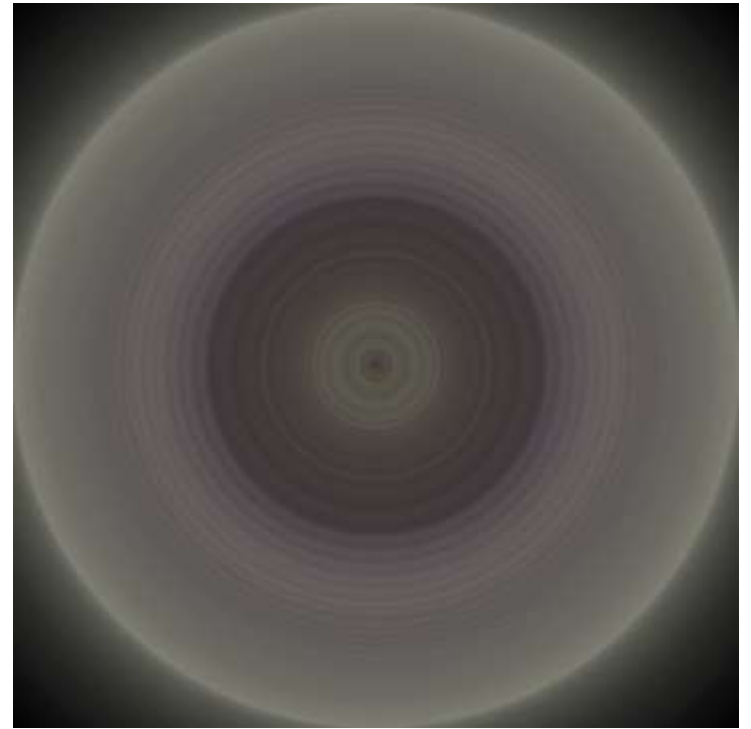
$$D = \{\mathbf{p} \mid \text{such that } |\mathbf{p}| \text{ is even}\}.$$

Invariants to circularly symmetric PSF



$$\mathcal{S} \equiv \mathcal{C}_\infty = \{h \in \mathcal{I} | h(r, \theta) = h(r)\}$$

Invariants to circularly symmetric PSF



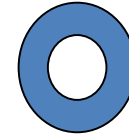
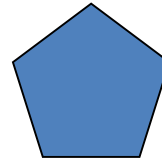
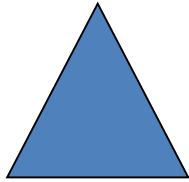
Invariants to circularly symmetric PSF

$$(P_{\infty} f)(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) d\theta$$

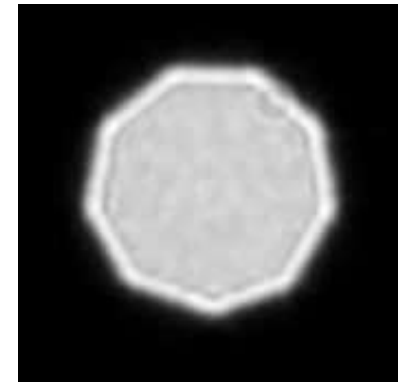
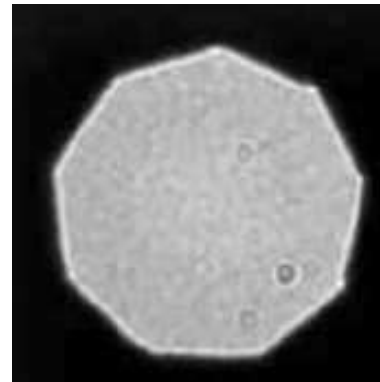
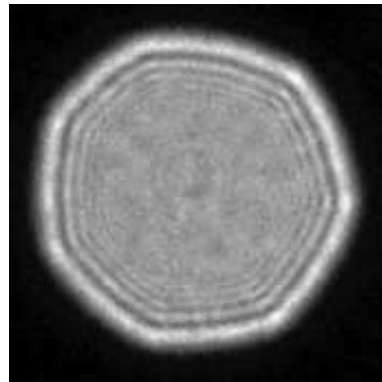
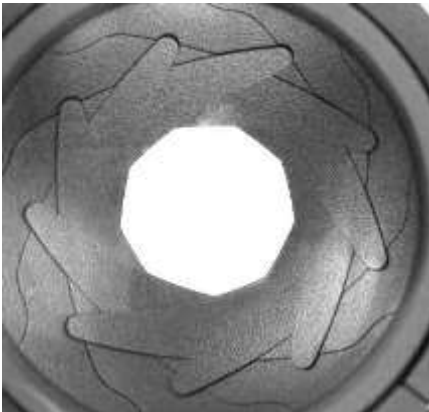
$$K_{\infty}(p, q)^{(f)} = c_{pq}^{(f)} - \frac{1}{c_{00}^{(f)}} \sum_{n=1}^q \binom{p}{n} \binom{q}{n} K_{\infty}(p-n, q-n)^{(f)} \cdot c_{nn}^{(f)}$$

$$D = \{(p, p) | p \geq 0\}$$

Invariants to N-fold rotational PSF

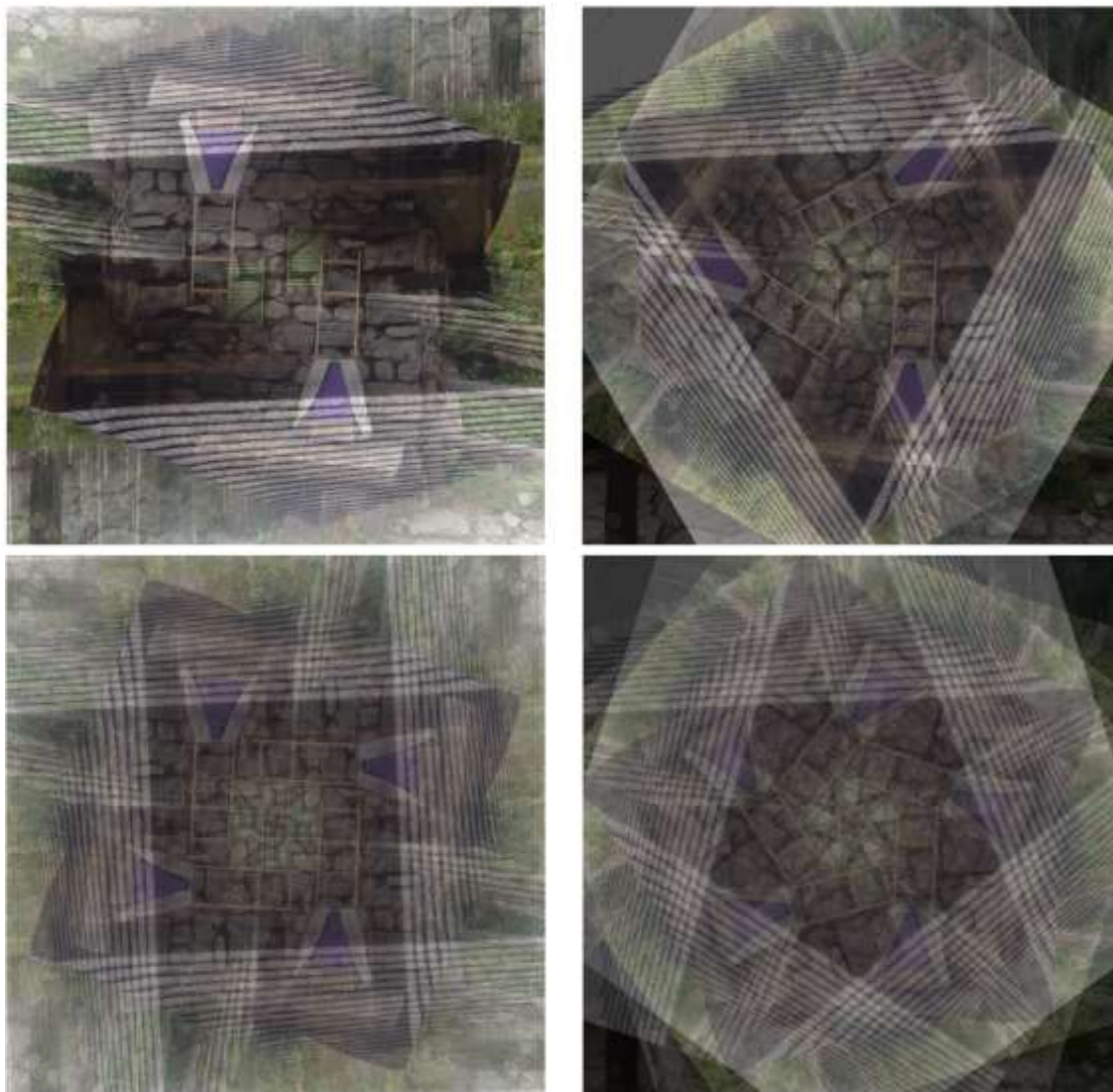


out-of-focus blur on a polygonal aperture
an aperture - physical diaphragm blades



$$S \equiv C_N = \{h \in \mathcal{I} \mid h(r, \theta) = h(r, \theta + 2\pi/N)\}$$

Invariants to N-fold rotational PSF



Invariants to N-fold rotational PSF

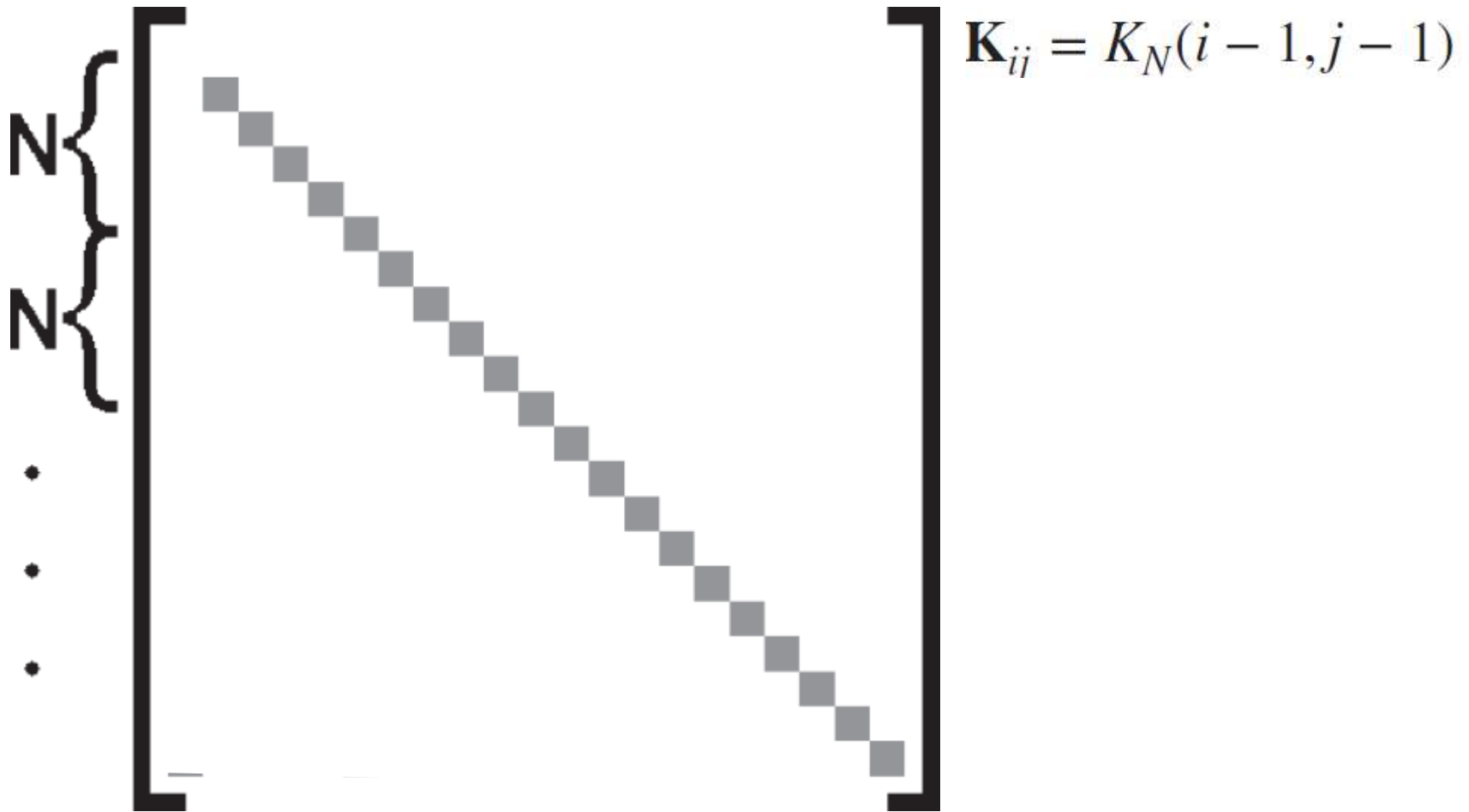
$$(P_N f)(r, \theta) = \frac{1}{N} \sum_{j=1}^N f(r, \theta + \alpha_j)$$

$$\alpha_j = 2\pi j/N$$

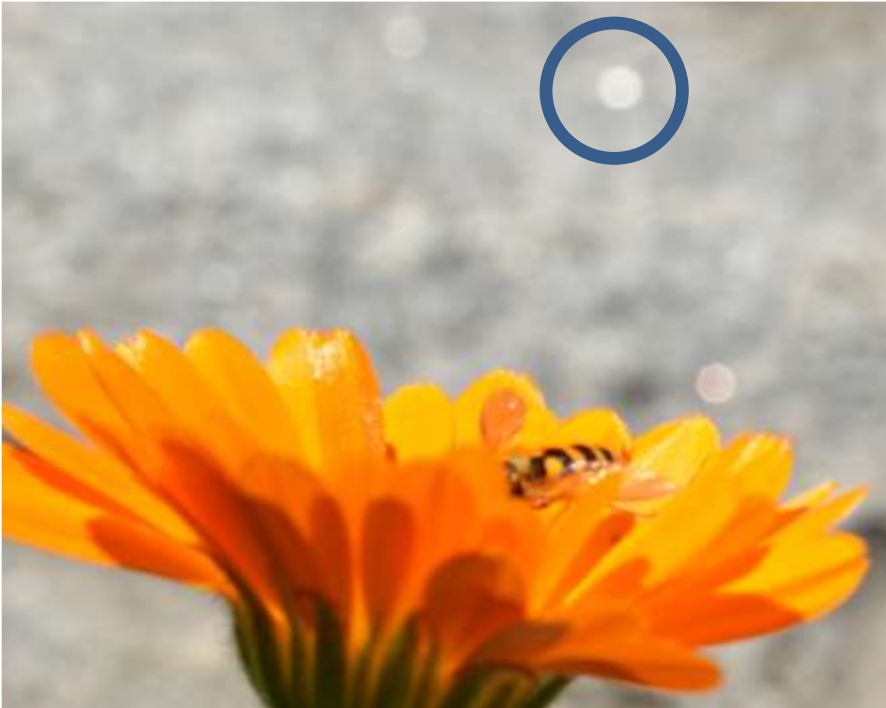
$$K_N(p, q)^{(f)} = c_{pq}^{(f)} - \frac{1}{c_{00}^{(f)}} \sum_{\substack{n=0 \\ 0 < n+m \\ (n-m)/N \text{ is integer}}}^p \sum_{m=0}^q \binom{p}{n} \binom{q}{m} K_N(p-n, q-m)^{(f)} \cdot c_{nm}^{(f)}$$

$$D_N = \{(p, q) | (p - q)/N \text{ is integer}\}$$

Invariants to N-fold rotational PSF



Invariants to N-fold rotational PSF



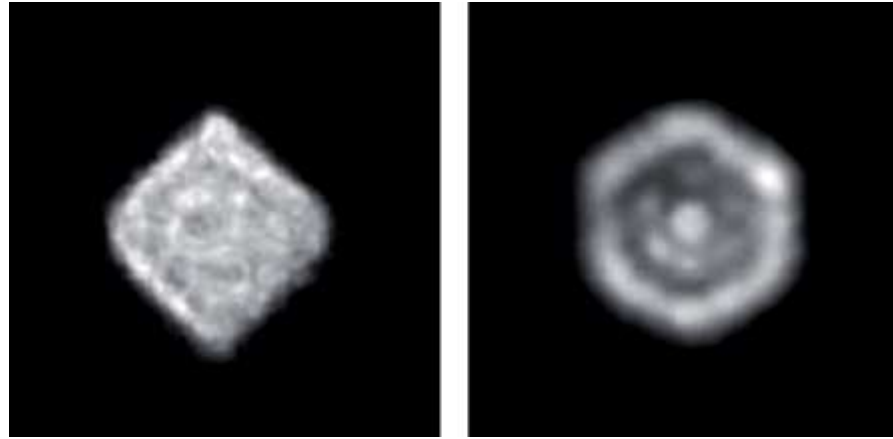
overestimate $N \rightarrow$ lose invariance

underestimate $N \rightarrow$ lose discriminability

different cameras \rightarrow greatest common divisor

Invariants to N-fold dihedral symmetry

PSF



N-FRS plus axial

$$f^\alpha(\mathbf{x}) = f(S\mathbf{x}) \quad S = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

$$\mathcal{S} \equiv D_N = \{h \in C_N \mid \exists \alpha \text{ such that } h(x, y) = h^\alpha(x, y)\}$$

$$\mathcal{S} \equiv C_N = \{h \in \mathcal{I} \mid h(r, \theta) = h(r, \theta + 2\pi/N)\}$$

Invariants to N-fold dihedral symmetry

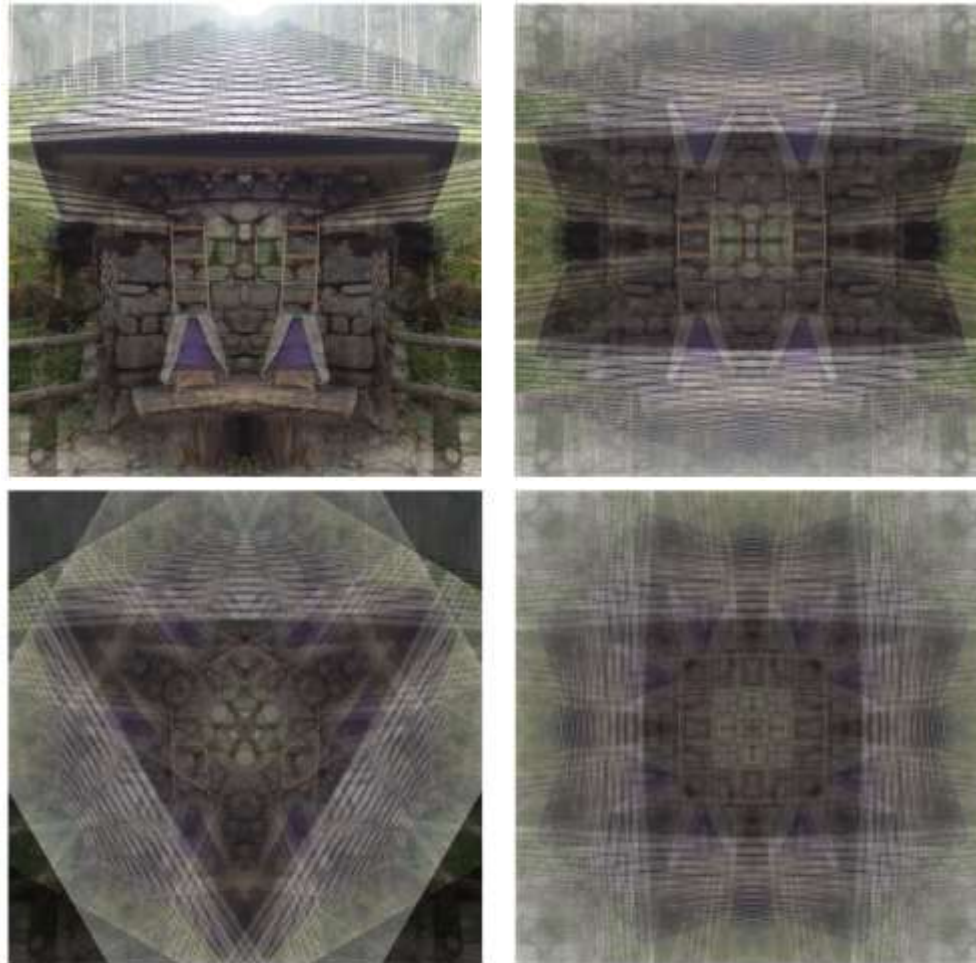
PSF

for any finite N , D_N is not closed under convolution

The closure property is preserved **only** if
the angle α is known and fixed for the whole S

Invariants to N-fold dihedral symmetry

PSF



Invariants to N-fold dihedral symmetry

PSF

$$Q_N^\alpha f = P_N((f + f^\alpha)/2)$$

$$L_N^\alpha(p, q) = c_{pq} - \frac{1}{2c_{00}} \sum_{\substack{j=0 \\ 0 < j+k \\ (j-k)/N \text{ is integer}}}^p \sum_{k=0}^q \binom{p}{j} \binom{q}{k} L_N^\alpha(p-j, q-k) \cdot (c_{jk} + c_{kj} e^{2i\alpha(p-q)})$$

Limitation - the axis orientation must be known and fixed

Invariants to directional blur PSF

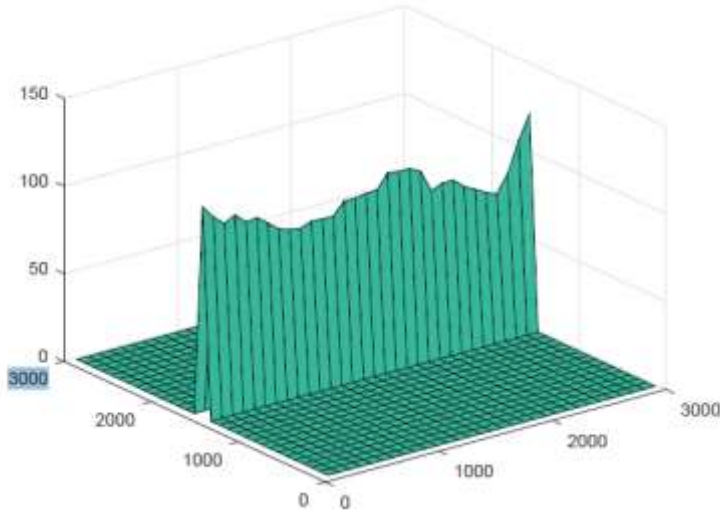
“motion” blur

$$h(x, y) = h_1(x)\delta(y).$$

S - set of all functions
of the given form



Invariants to directional blur PSF



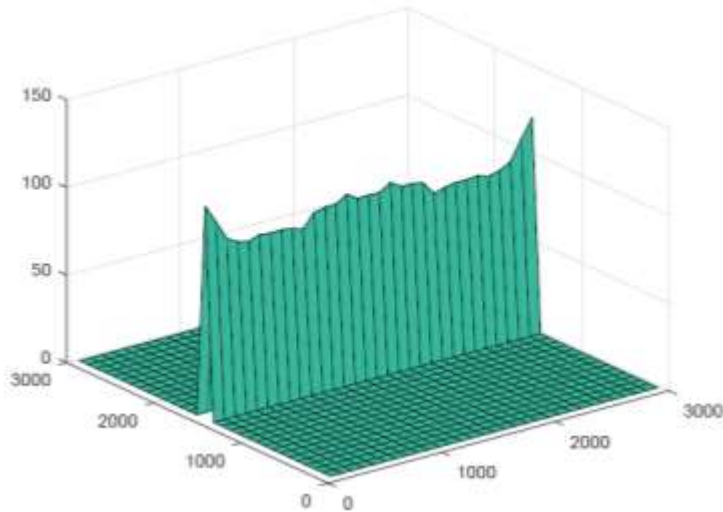
$$(Pf)(x, y) = \delta(y) \int f(x, y) dy$$

$$D = \{(p, q) | q = 0\}$$

$$M(p, q) = m_{pq} - \frac{1}{m_{00}} \sum_{k=1}^p \binom{p}{k} M(p-k, q) m_{k,0}$$

“motion” blur – direction has to be known

Invariants to directional blur PSF



$$h_1(-x) = h_1(x)$$

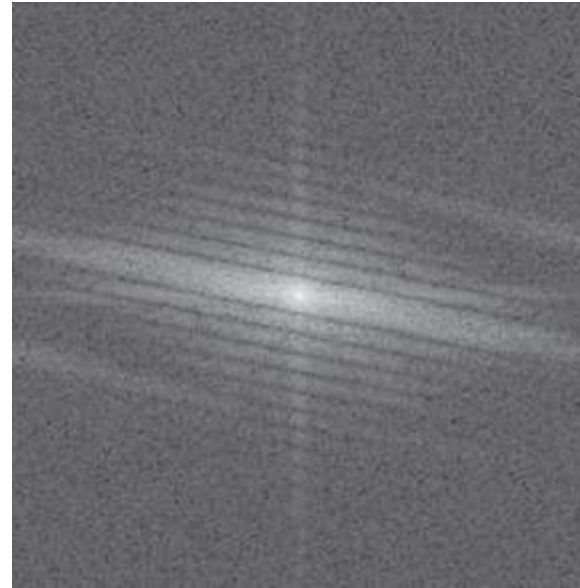
+ symmetry assumption

$$D^s = \{(p, q) | p = 2k, q = 0\}$$

$$(P^s f)(x, y) = \delta(y) \frac{1}{2} \int (f(-x, y) + f(x, y)) dy$$

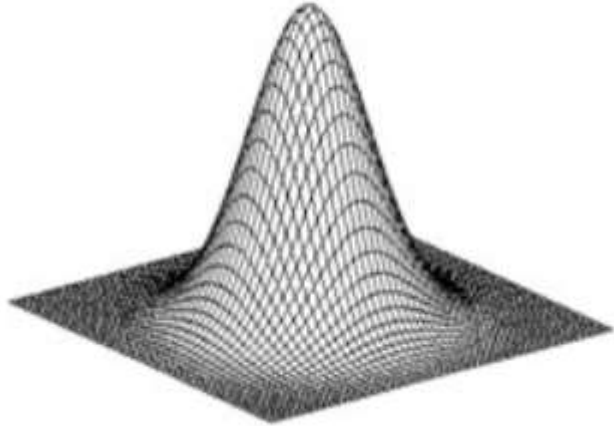
$$M^s(p, q) = m_{pq} - \frac{1}{m_{00}} \sum_{\substack{k=2 \\ k \text{ even}}}^p \binom{p}{k} M^s(p-k, q) m_{k,0}$$

Invariants to directional blur PSF





Invariants to Gaussian PSF



$$G(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$S_G = \{aG_\sigma | \sigma \geq 0, a \neq 0\}$$

- symmetric - special case of circular symmetry blur
- asymmetric – special case of 2-fold dihedral PSF
- designed – higher discriminability

- not “image function” – no bounded support
- set of Gaussians not a vector space

Invariants to Gaussian PSF



circularly symmetric 2D Gaussian

$$c_{pq}^{(G_\sigma)} = \begin{cases} (2\sigma^2)^p p! & p = q \\ 0 & p \neq q \end{cases}$$

$$P_G f(x, y) = c_{00}^{(f)} G_s(x, y) \equiv \frac{c_{00}^{(f)}}{2\pi s^2} e^{-\frac{x^2+y^2}{2s^2}}$$

$$s^2 = c_{11}^{(f)} / 2c_{00}^{(f)}$$

$P_G f$ has the same c_{00} and c_{11} as f

Invariants to Gaussian PSF

$$K_G(p, q)^{(f)} = c_{pq}^{(f)} - \sum_{k=1}^q k! \binom{p}{k} \binom{q}{k} \left(\frac{c_{11}^{(f)}}{c_{00}^{(f)}} \right)^k K_G(p-k, q-k)^{(f)}$$

$$K_G(p, q)^{(f)} = \sum_{j=0}^q j! \binom{p}{j} \binom{q}{j} \left(-\frac{c_{11}^{(f)}}{c_{00}^{(f)}} \right)^j c_{p-j, q-j}^{(f)}$$

If $p = q = 1 \rightarrow$ the invariant is zero

Combined moment invariants

set S of the admissible PSFs - closed also w.r.t. the considered geometric transformations

N -FRS blur ($N > 2$) + affine transform

dihedral blur + general rotation

Gaussian radially symmetric blur + affine transform

directional blur + rotation

any blur + translation and/or uniform scaling

Combined moment invariants rotation

N -FRS blur, based on complex moments

radial Gaussian blur, based on complex moments

$$K'(p, q) = e^{-i(p-q)\alpha} \cdot K(p, q)$$

$$I = \prod_{j=1}^n K(p_j, q_j)^{k_j} \quad \sum_{j=1}^n k_j(p_j - q_j) = 0$$

$$K(p, q)K(1, 2)^{p-q}$$

Combined moment invariants dihedral

restrict the admissible rotation angles to integer multiples of π/N

$$L_N^\alpha(p, q)' = e^{-i(p-q)\theta} \cdot L_N^\alpha(p, q)$$

Invariants to convolution and affine transform



centrosymmetric PSF
other not closed under AT

Invariants to convolution and affine transform



$$I(\mu_{00}, \dots, \mu_{PQ})$$
$$C(p, q)$$

affine moment invariant
blur invariant

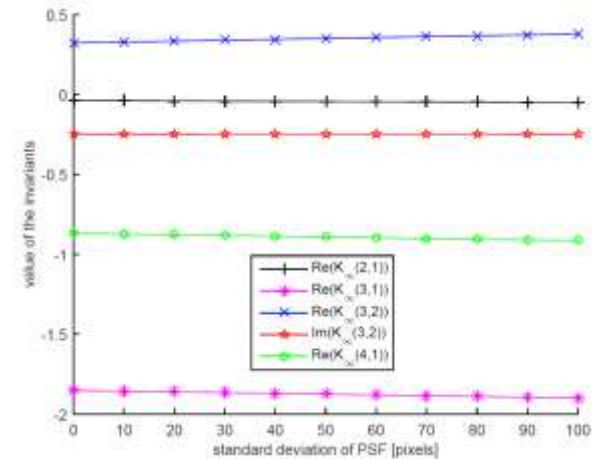
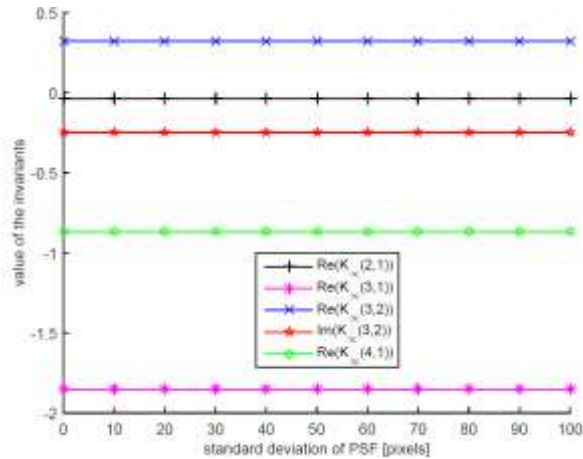
$$I(C(0,0), \dots, C(P,Q))$$

combined blur-affine invariant
CBAI

Reported applications of convolution and combined invariants

- Character/digit/symbol recognition in the presence of vibration, linear motion or out-of-focus blur
- Robust image registration (medical, satellite, ...)
- Detection of image forgeries

Robustness of blur invariants



Leaf recognition system MEW2010

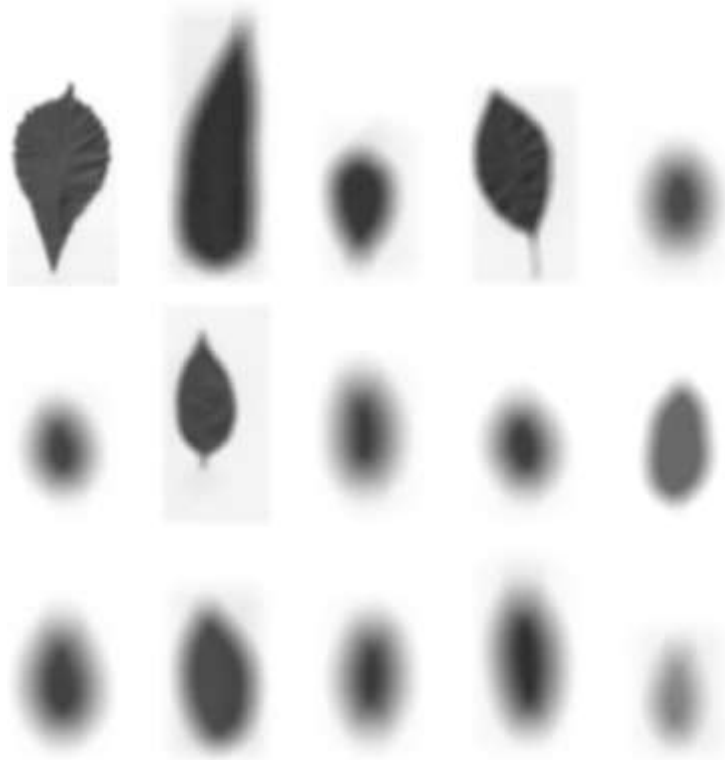
10 000 tree leaves

100 classes (species)

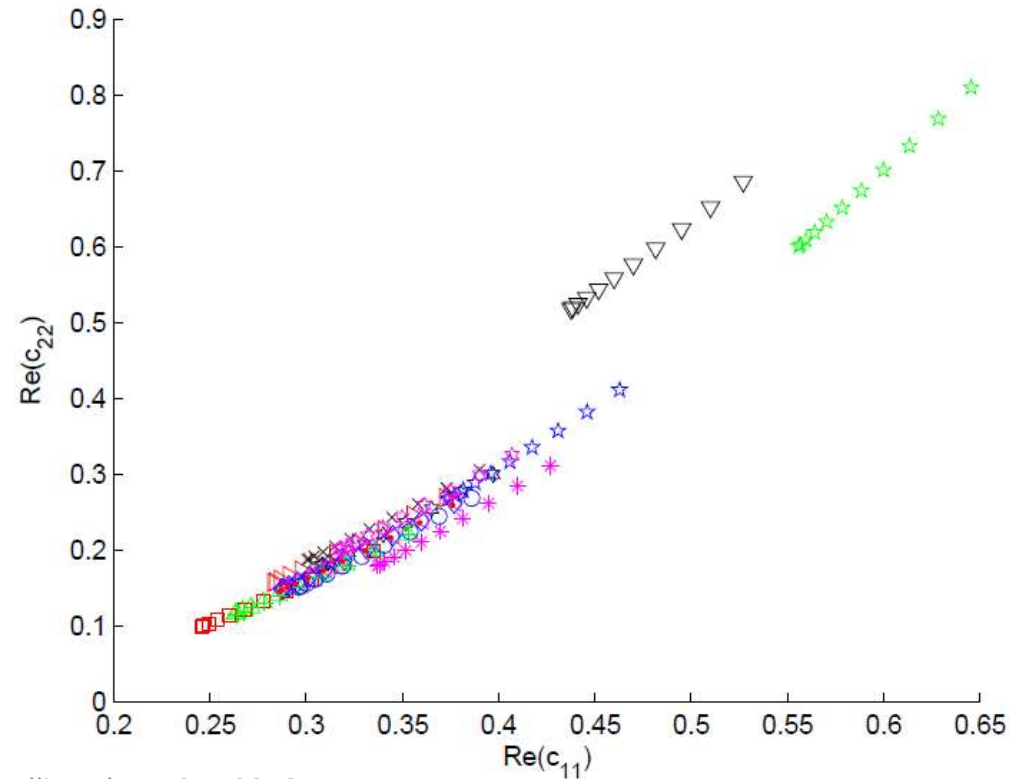
Recognition based solely on the contour



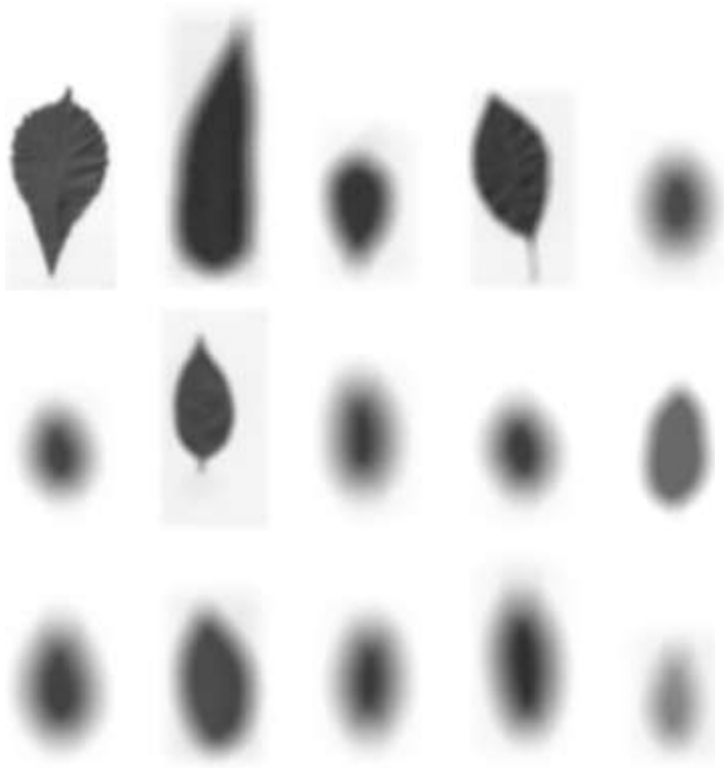
Leaf recognition system MEW2010



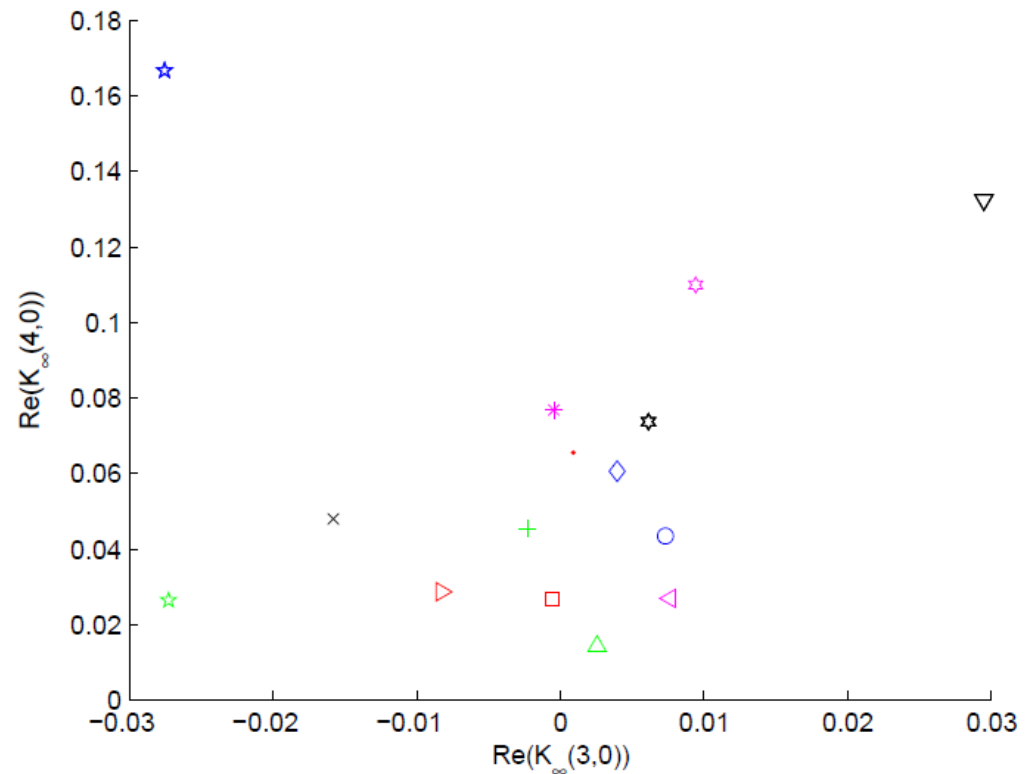
Plain moments



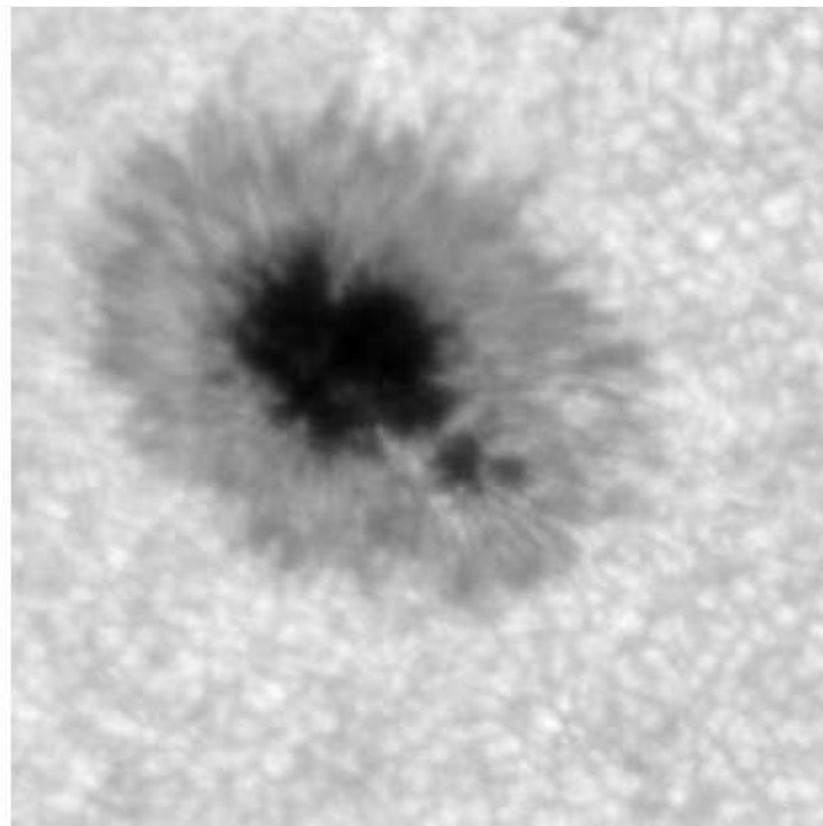
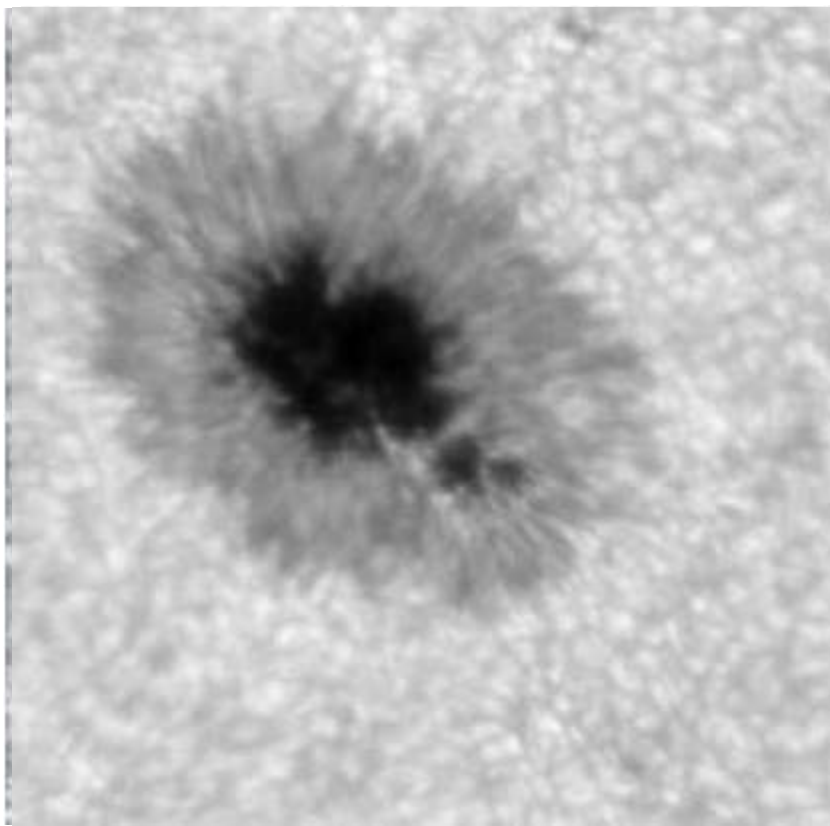
Leaf recognition system MEW2010



Invariants



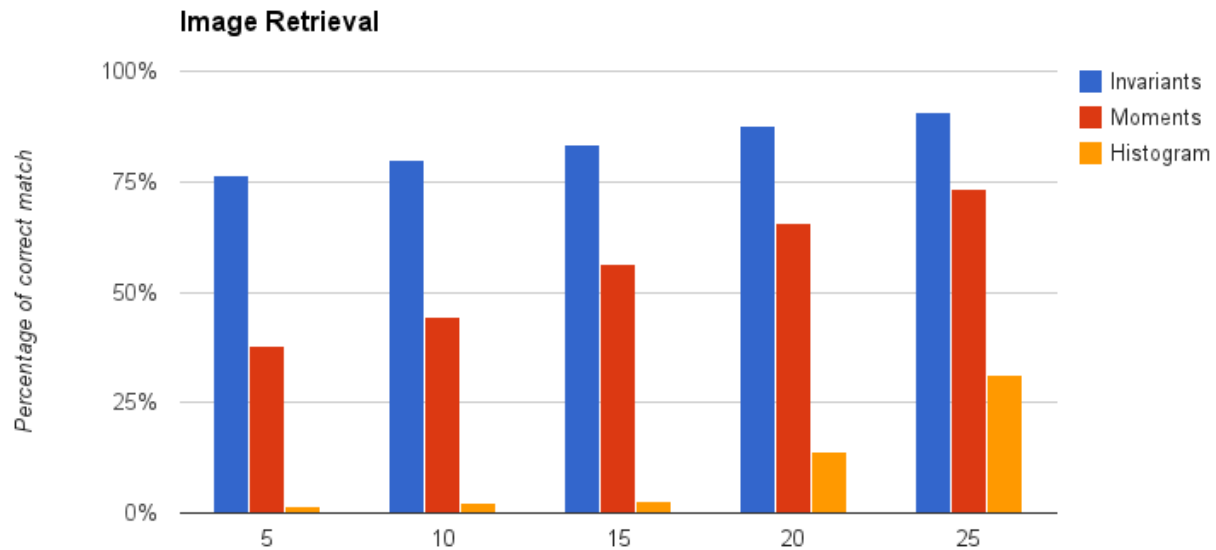
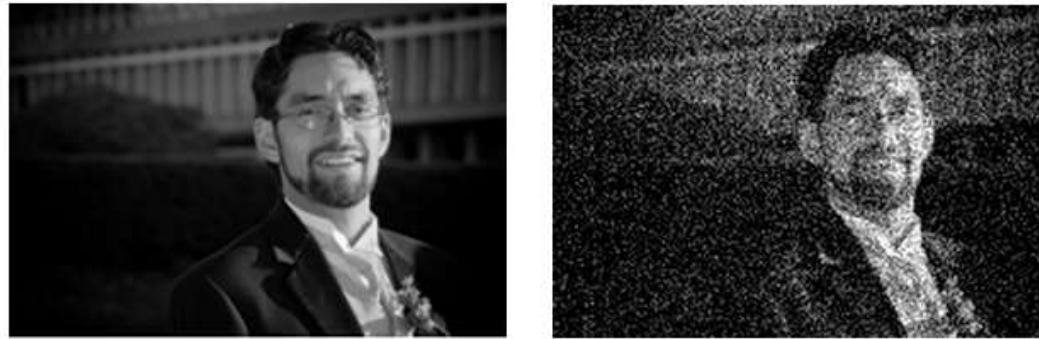
Recognition abilities of blur invariants



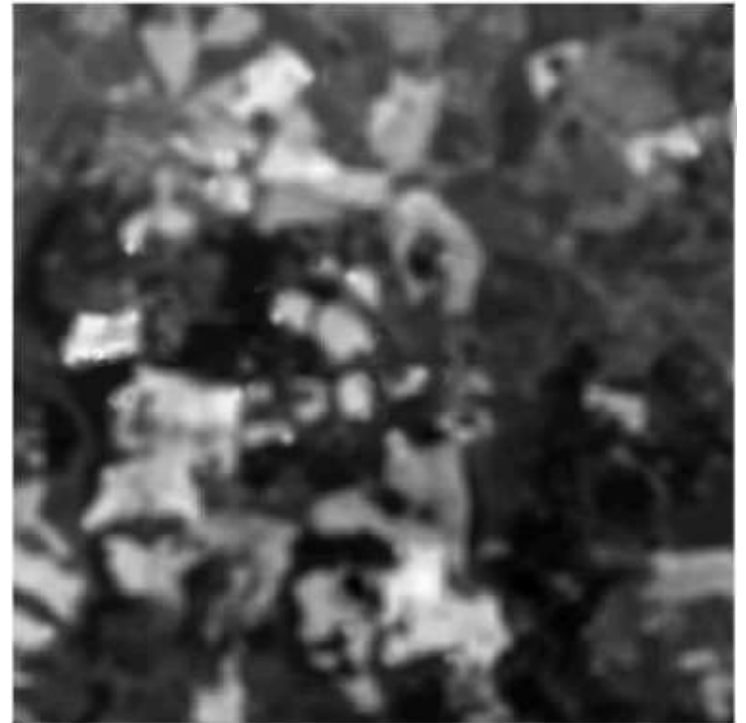
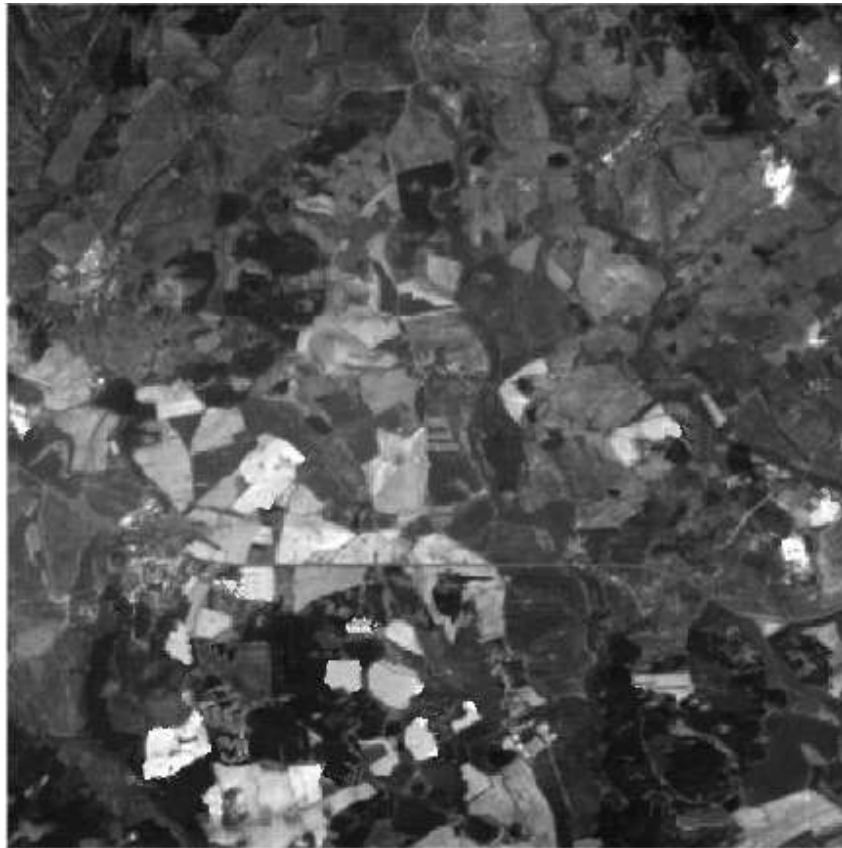
Recognition abilities of blur invariants

Image	Mean error	Standard deviation
	Cross-correlation	
(b)	7.30	3.82
(c)	7.29	3.81
(d)	7.28	3.80
	Gaussian blur invariants	
(b)	0.88	0.45
(c)	0.90	0.47
(d)	0.85	0.44

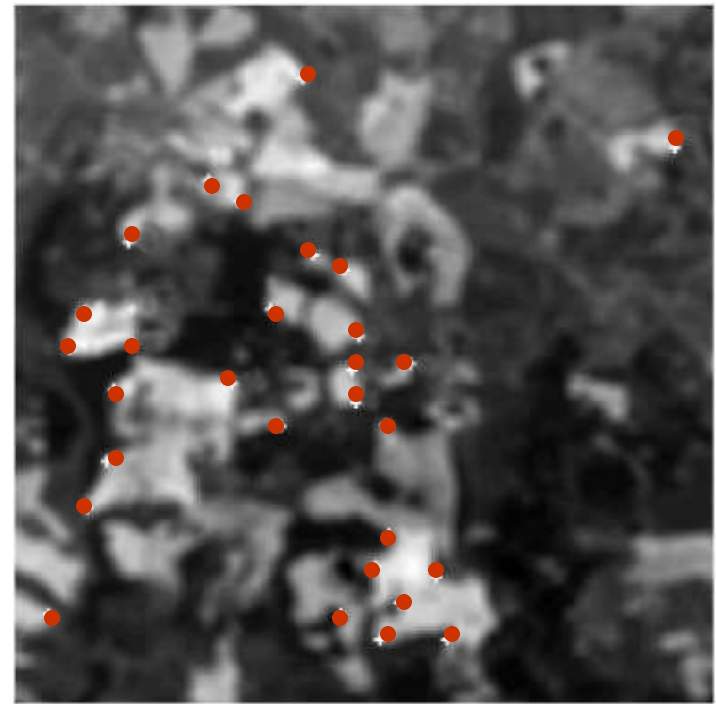
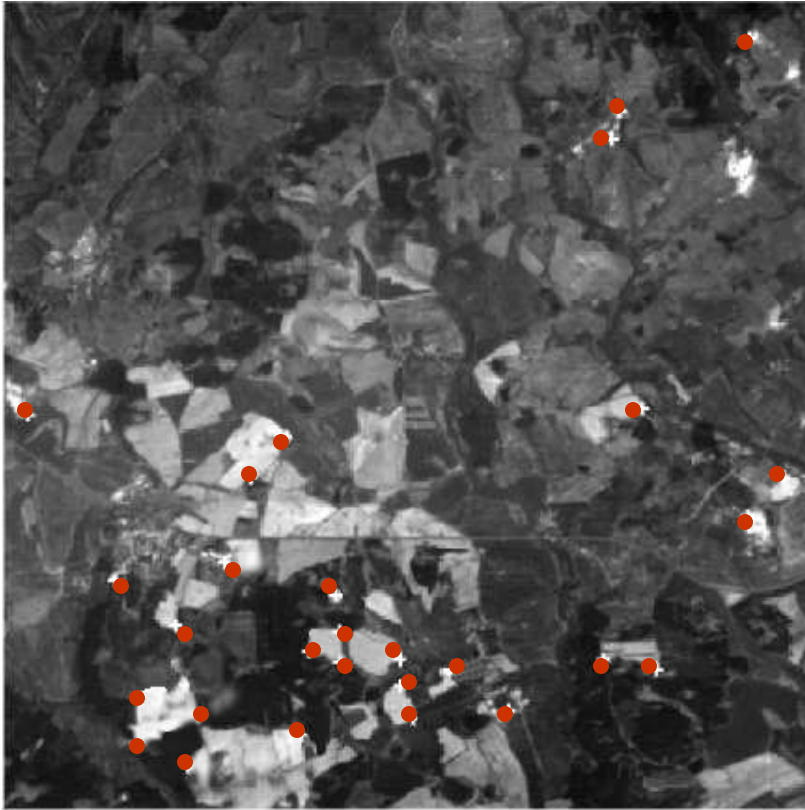
Blur invariants and CBIR



Satellite image registration by combined invariants

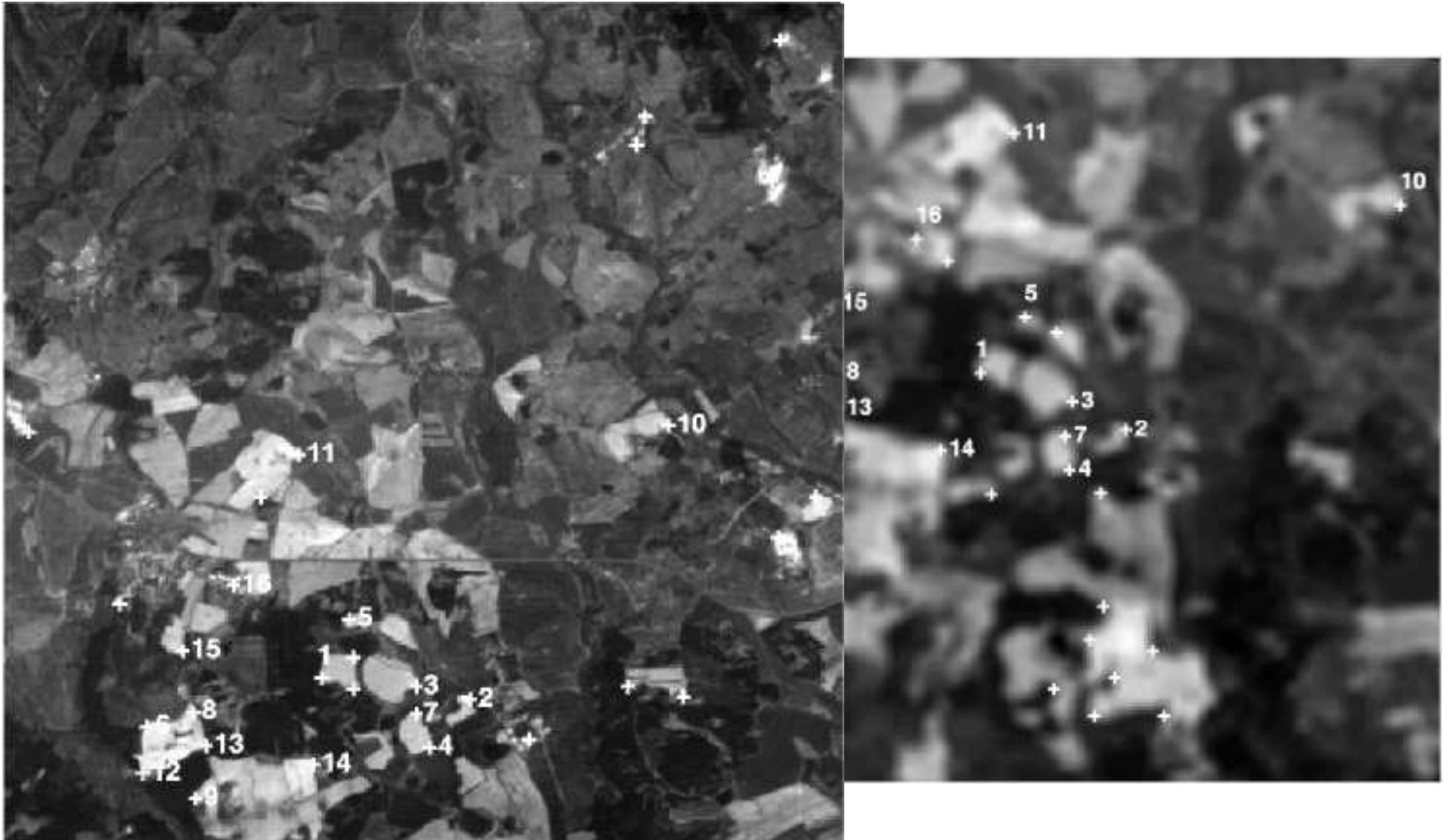


Control point detection

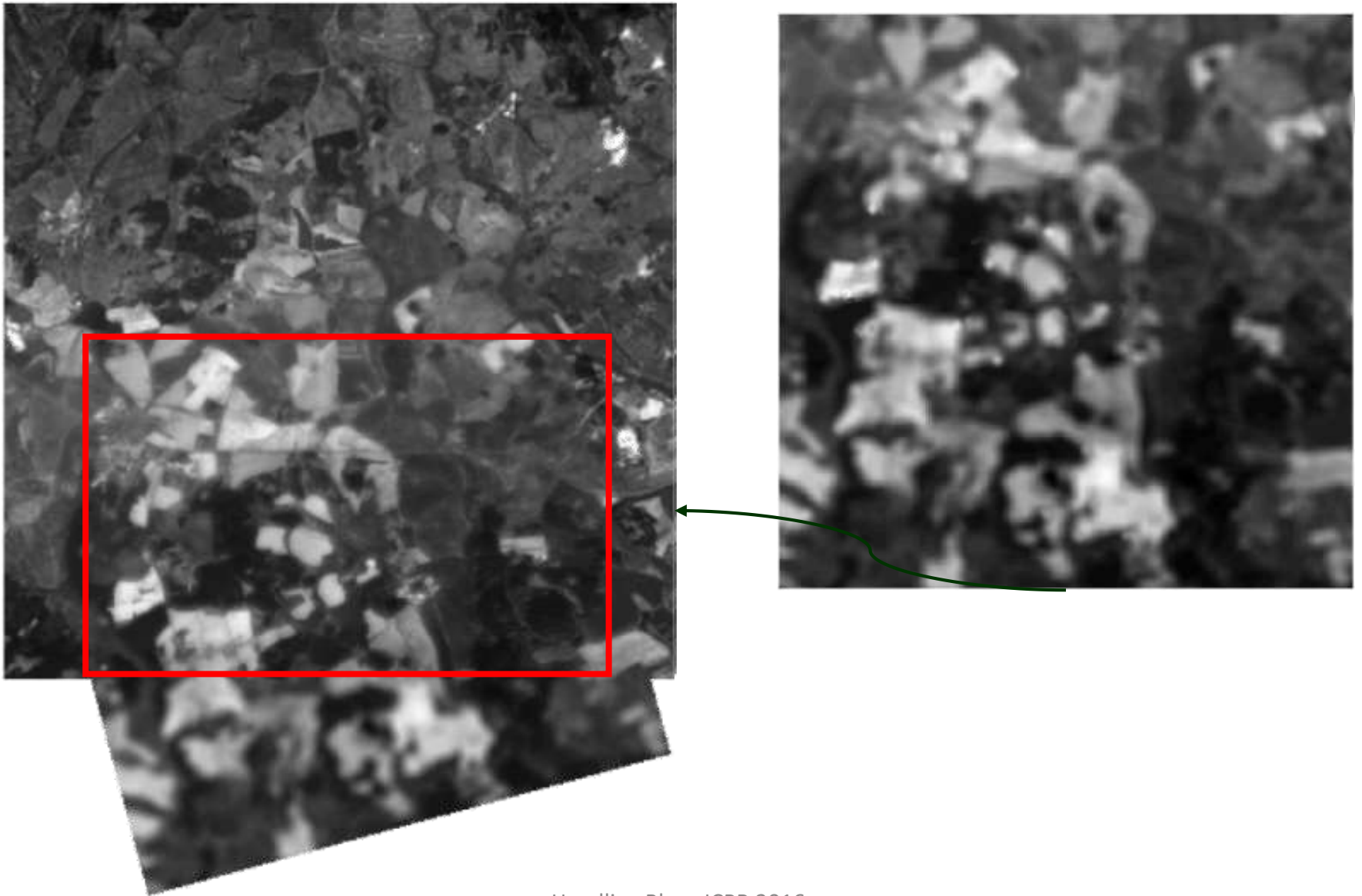


$$\min_{k,m} \text{distance}((v1_k, v2_k, v3_k, \dots), (v1_m, v2_m, v3_m, \dots))$$

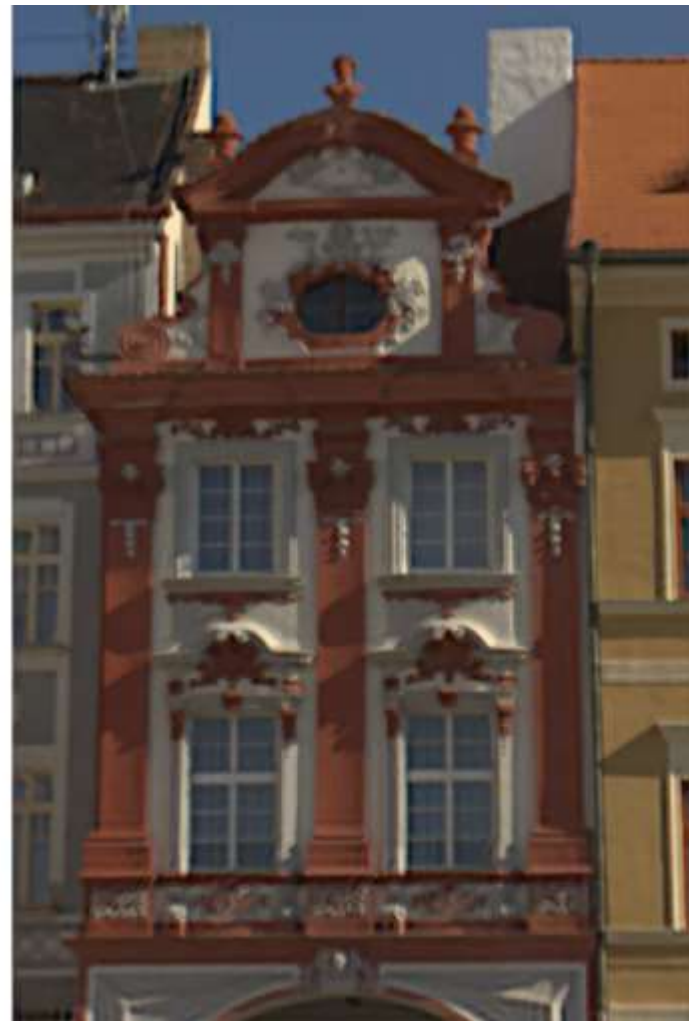
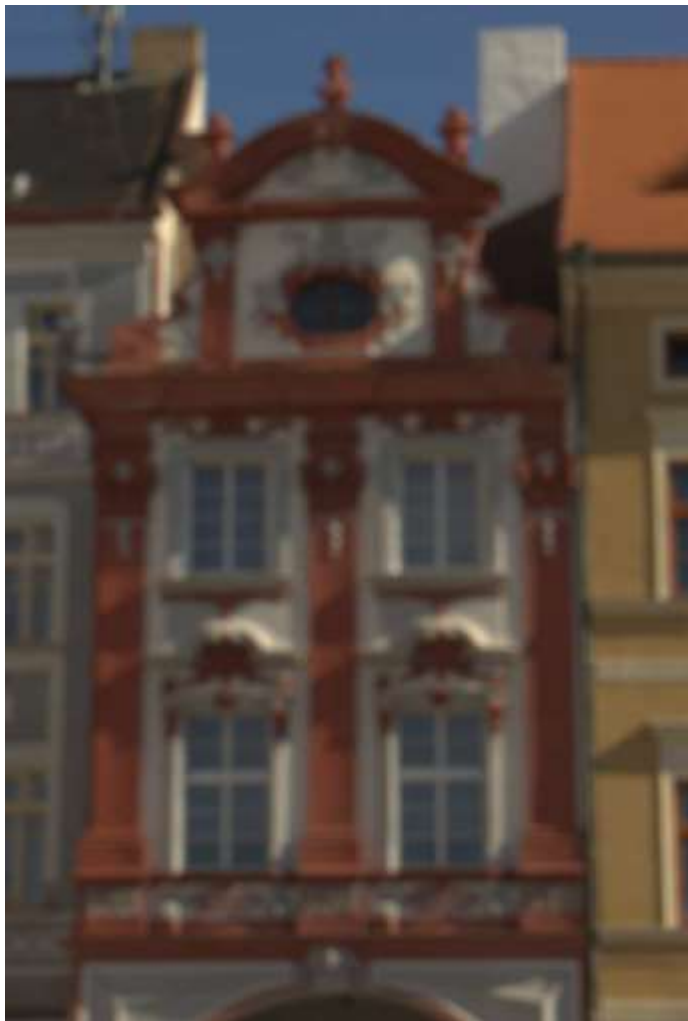
Control point matching



Registration result



Multichannel blind deconvolution



Multichannel blind deconvolution

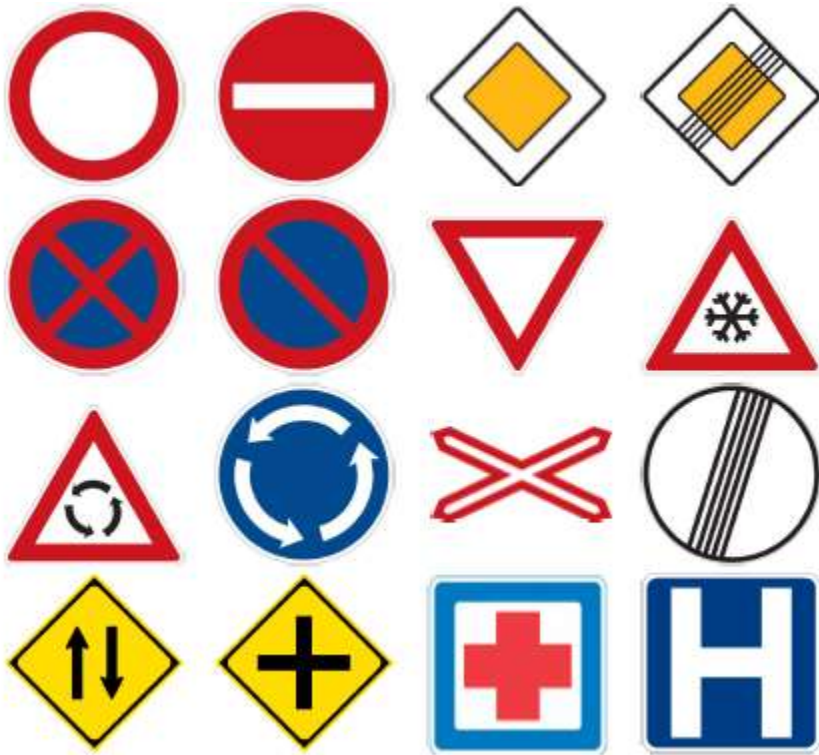


The Poor Fisherman, Paul Gauguin, 1896

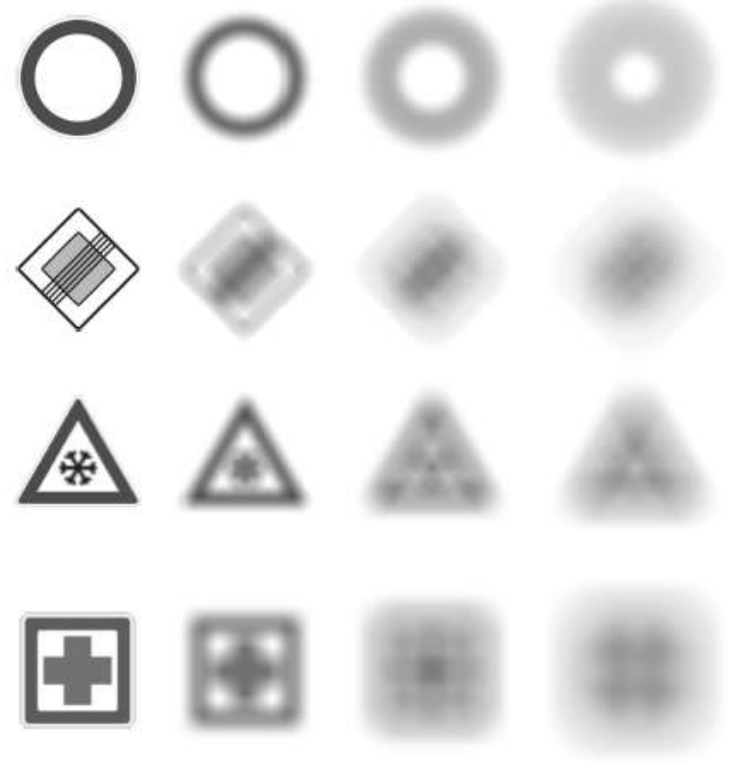
Detecting forgeries



Recognition of symmetric objects



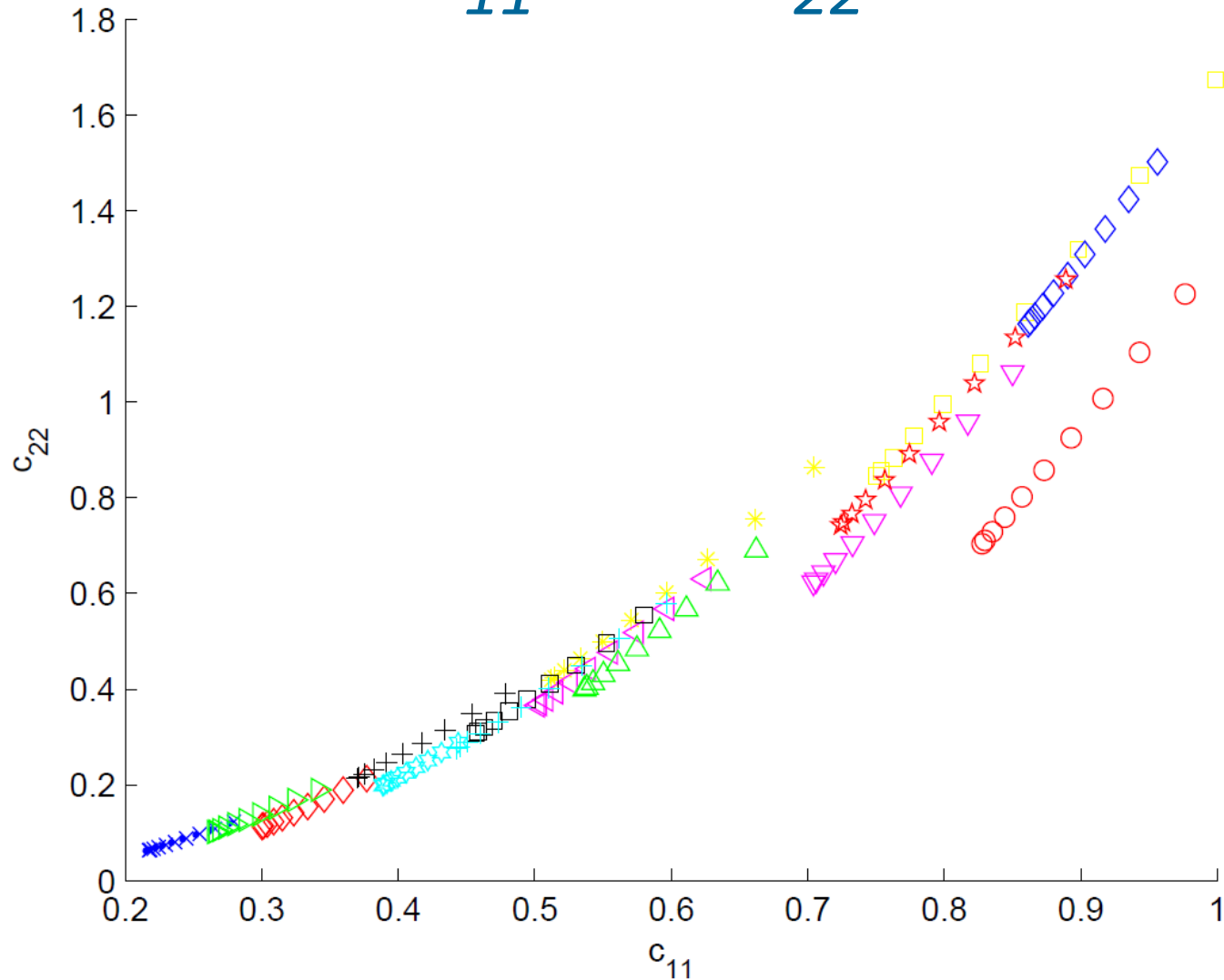
7 signs of 2-FRS
4 signs of 4-FRS



4 signs of 3-FRS
1 sign having ∞ -FRS

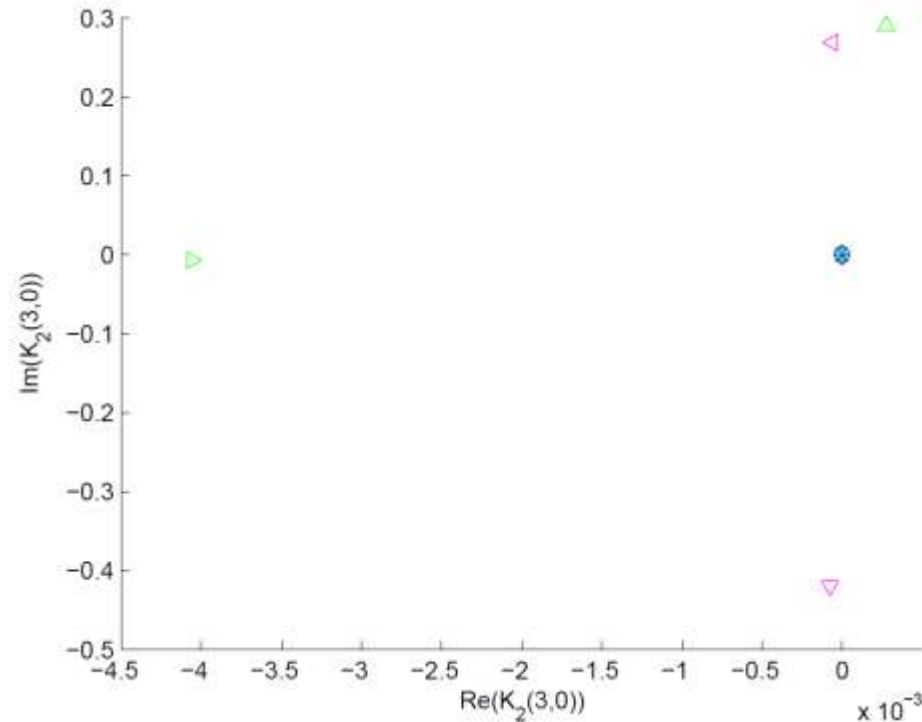
Recognition of symmetric objects

c_{11} and c_{22}



Recognition of symmetric objects

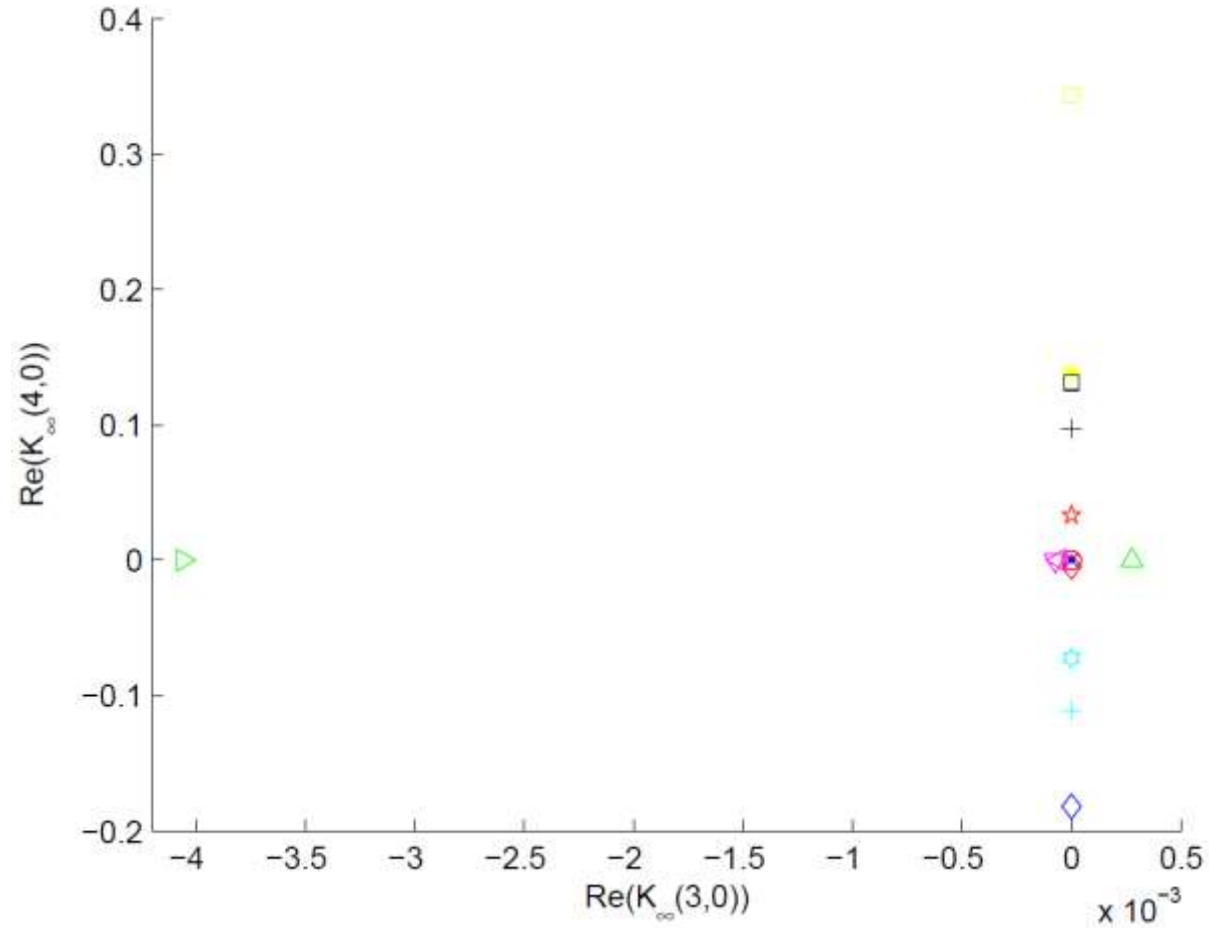
Real and Im of $K_2(3,0)$



signs with even-fold symmetry lie in its nullspace

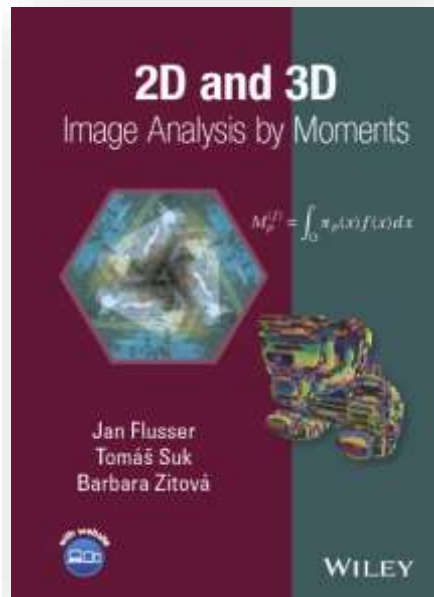
Recognition of symmetric objects

Real and Im of $K_\infty(3,0)$



Reconstruction versus Invariants

- Whole image is needed - >
reconstruction
- Scene analysis, object detection ->
invariants



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Thank you...